THE EFFECT OF UNCERTAINTY ON LANCHESTER-TYPE EQUATIONS OF COMBAT

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ON
LANCHESTER-TYPE EQUATIONS OF COMBAT

by

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The Effect of Uncertainty
on
Lanchester-Type Equations of Combat

by

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LIST OF SYMBOLS

M	=	Random variable of the number of survivors for the X force in a stochastic model.
N	=	Random variable of the number of survivors for the Y force in a stochastic model.
$x(t)$	=	Number of survivors of X force at time t for a deterministic model.
$y(t)$	=	Number of survivors of Y force at time t for a deterministic model.
$m(t)$	=	Realization of M at time t.
$n(t)$	=	Realization of N at time t.
m_o	=	Initial size of X force.
n_o	=	Initial size of Y force.
F_X	=	Ratio of the number of casualties to the initial force size of the X force at which the X force disengages.
F_Y	=	Ratio of the number of casualties to the initial force size of the Y force at which the Y force disengages.
m_{bp}	=	Largest integer part of $(1-F_X)\cdot m_o$. The number of survivors at which the X force disengages.
n_{bp}	=	Largest integer part of $(1-F_Y)\cdot n_o$. The number of survivors at which the Y force disengages.
$F(t,x,y)$	=	Rate of attrition of the X force for a deterministic model.
$G(t,x,y)$	=	Rate of attrition of the Y force for a deterministic model.
a	=	Rate at which a Y combatant attrits the X force.
b	=	Rate at which a X combatant attrits the Y force.

$F(t,m,n)dt$	=	Probability that in the small time interval dt, an X combatant is destroyed, the number of fighting units existing at the time t preceeding that interval being m and n; for a stochastic model.
$G(t,m,n)dt$	=	Probability that in the small time interval dt, a Y combatant is destroyed, the number of fighting units existing at the time t preceeding that interval being m and n; for a stochastic model.
$A(m,n)$	=	Special case of $F(t,m,n)$ with constant coefficients.
$B(m,n)$	=	Special case of $G(t,m,n)$ with constant coefficients.
$\bar{m}(t)$	=	Expected value of M at time t.
$\bar{n}(t)$	=	Expected value of N at time t.
$V(M,t)$	=	Variance of M at time t.
$V(N,t)$	=	Variance of N at time t.
$P(t,m,n)$	=	Probability that at time t, X has m survivors and Y has n survivors, given that at time $t = 0$, $m = m_o$ and $n = n_o$.
P_X	=	Probability X wins in a stochastic model.
P_Y	=	Probability Y wins in a stochastic model.
$P(m,n;r,s)$	=	Probability that the X force has m survivors and the Y force has n survivors given that they started with r combatants and s combatants, respectively.
$\Delta_X(t)$	=	$\bar{m}(t) - x(t)$
$\Delta_Y(t)$	=	$\bar{n}(t) - y(t)$
$\Delta\%_X(t)$	=	bias for the X forces of the two models at time t.
$\Delta\%_Y(t)$	=	Percent of bias for the Y forces of the two models at time t.

I. INTRODUCTION

Although combat is a complex random process, analysts frequently model it with deterministic Lanchester-type equations for reasons of mathematical tractability and computational convenience. Such deterministic models are being used extensively today in defense planning studies. However, one can include the various random aspects of combat (e.g., uncertainty in enemy force level, random occurrence of casualties, etc.) in models and develop so-called stochastic models of the combat process. Thus, a basic way of classifying a combat model is whether the model is a deterministic model or a stochastic one.

Moreover, there are fundamental differences between deterministic and stochastic models. Deterministic dynamic combat models predict the future with certainty; for given initial conditions there is no question about what the state of the conflict will be at any future time. On the other hand, stochastic dynamic models only tell one the "chances" of what will happen. With a stochastic model one does not know the future with certainty. That is, it cannot be guaranteed that a specific state will be reached in the future — there is only a "probability" it will be reached.

Each type of model has inherent advantages and disadvantages which are partially summarized in Table I.

Deterministic Models

Advantages	Disadvantages
1. Requires one solution of the model for a useful outcome	1. A further abstraction from reality
2. Less expensive to run	
3. Interactions are more easily analyzed	

Stochastic Models

Advantages	Disadvantages
1. Closer to reality	1. Requires many replications to provide statistically significant results
	2. More expensive to run
	3. Interactions are much more difficult to analyze

TABLE I. Advantages and Disadvantages of Deterministic and Stochastic Models

The solutions to stochastic models quickly become quite complex as the number of combatants increases, and are not very useful; so one must use other methods to obtain insights into the dynamics of combat. The two most commonly used methods, Monte Carlo simulation and finite difference approximation, both possess certain disadvantages. Monte Carlo combat simulations require that the battle be replicated many times in order to obtain a good statistical estimate of the probable course of battle, while finite differen~~t~~ce approximation methods usually require digital computer implementation (including development of a computer program).

On the other hand, solutions to deterministic models can sometimes be useful for developing insights into the dynamics of combat. Even when they are not, the model must be solved only once by finite difference approximation as opposed to the many replications required for Monte Carlo simulation. Also, finite difference approximations of deterministic models are generally less costly in terms of time, and manhours required to implement the computer program, than the finite difference approximations of stochastic models. The economic costs of using the models then, is the basic reason most models of combat used in the past have been deterministic. But can the complex random process of combat be adequately represented by a deterministic model? Does one destroy any of the essential features of the random combat process by considering a deterministic model as representing the "average" course of combat?

The objective of this thesis is to consider this question by investigating the differences in results obtained from a deterministic and a stochastic model of combat.

The importance of this objective stems from the fact that both types of models are used extensively today in the decision making process in the U.S. Army. For example, the Dynamic Tactical Simulation, DYNTACS-X, and the Battalion Differential Model, BLDM, widely known as the Bonder/IUA model, are two models that simulate battalion level mid-intensity armored combat. DYNTACS is an event sequenced stochastic model that is being or has been used by the Rock Island Arsenal to evaluate proposed mobility improvements to the M60 tank, the family of scatterable mines concept, cannon launched guided projectiles, remotely piloted vehicles, and the XM1 tank. Bonder/IUA is a deterministic Lanchester-type model that is being or has been used by the Rock Island Arsenal in sensitivity analyses and evaluation of the MBT70 study, several anti-armor automatic cannon concepts, and the Low Dispersion Automatic Cannon study. A study done by the Rock Island Arsenal [Ref. 5] shows that there are differences in the results of the two models with "equivalent" inputs. With the present state of the art, it is not possible to predict with any accuracy when significant differences between the results of the models will occur. It is felt that this thesis may provide trends which will allow a user to gain insights into these differences and more accurately determine when a deterministic model represents combat "satisfactorily".

Models have had wide application in the military decision making process, and the realm of possible future applications is even larger. For example, the Engineer Strategic Studies Group [Ref. 10] has, as its name implies, evaluated several aspects of military engineering, but with little emphasis on the use of combat models. Much of this has been caused by a dearth of models that could be applied to engineer studies. However, this is being altered; for example, the U.S. Army Concepts Analysis Agency [Ref. 30] has accepted as a future tool a model developed by ESSG, the Fortification and Obstacle Effect Simulation, 1975. This is a high resolution model used in analyzing obstacle effectiveness. The model is part of a research effort by ESSG to design a method for assessing the contribution that obstacles make to combat operations. This model was developed using ideas from other models, but it was based on the identification of specific objectives. From these objectives measures of effectiveness were developed, and from these measures the model was developed. It is possible, with the identification of proper objectives, to develop models for other phases of military engineering and, more generally, for the evaluation of other phases of combat. It is therefore felt that combat models do, and will continue to have an ever increasing application to the evaluation of military systems.

As was noted, however, it is not always clear which type of model should be applied — a stochastic or a deterministic model. From an economic standpoint, the advantages of the

deterministic model weight heavily in its favor. If the deterministic results do not significantly differ from those of the stochastic model, it seems the former should be used. The question is, when are the models "similar"?

II. LITERATURE SURVEY

This chapter briefly reviews Lanchester's well-known equations of combat and some simple extensions, with the discussion focused on stochastic formulations of the combat attrition process. The review consists of three parts; (A) Deterministic Models, (B) Stochastic Models, and (C) Comparisons of the two types.

A. DETERMINISTIC MODELS

In 1914 Lanchester [Ref. 19] hypothesized that under conditions of "modern warfare" combat between two homogeneous forces could be described by the equations

$$\frac{dx}{dt} = -\ ay \qquad (1)$$

$$\frac{dy}{dt} = -\ bx \qquad (2)$$

with initial conditions

$$x(t = 0) = x_o \qquad (3)$$

$$y(t = 0) = y_o \qquad (4)$$

where a and b are commonly referred to as the Lanchester attrition rate coefficients and $x(t)$ and $y(t)$ are force levels. One set of assumptions that has been hypothesized

6

(proposed by H. Weiss [Ref. 35] and cited by Dolansky
7
[Ref. 9]) to yield equations (1) and (2) is

1. Two forces are engaged in a battle. The units are homogeneous, but the rate of attrition may be different for each force.

2. Each firer on either side is within the range of all targets on the other side.

3. Attrition rate coefficients are constant.

4(a). Firers have perfect knowledge of the target locations and disposition so that they fire only at live targets, and fire can be shifted instantaneously.

5(a). Fire is uniformly distributed over remaining targets.

The state solution (time "independent" solution relating the force levels) is given by

$$b(x_o^2 - x(t)^2) = a(y_o^2 - y(t)^2) \qquad (5)$$

hence the name "square-law" attrition process.

The time solution for equations (1) and (2) is given by (see Morse and Kimball [Ref. 22])

$$x(t) = x_o \cosh(\sqrt{ab}\, t) - \sqrt{a/b}\, y_o \sinh(\sqrt{ab}\, t) \qquad (6)$$

$$y(t) = y_o \cosh(\sqrt{ab}\, t) - \sqrt{b/a}\, x_o \sinh(\sqrt{ab}\, t) \qquad (7)$$

The square-law was developed as a by-product of the original work by Lanchester to quantitatively justify the principle of concentration under certain combat conditions.

Lanchester was contrasting these conditions with the conditions of "ancient warfare" between two homogeneous forces. He postulated that the latter could be described by the equations

$$\frac{dx}{dt} = - axy \tag{8}$$

$$\frac{dy}{dt} = - bxy \tag{9}$$

with initial conditions

$$x(t = 0) = x_o \tag{10}$$

$$y(t = 0) = y_o \tag{11}$$

One set of assumptions that has been hypothesized (also proposed by H. Weiss [Ref. 9 ~~36~~] and cited by Dolansky [Ref. 9]) to yield equations (10) and (11) are very similar to the assumptions listed previously for the square-law. The first three assumptions are identical to the first three square-law assumptions.

4(b). Each firer knows the area that contains targets, but does not know exact target locations or the consequences of his fire.

5(b). Fire from surviving units is distributed uniformly over the target area.

The state solution is given by

$$b(x_o - x(t)) = a(y_o - y(t)) \tag{12}$$

hence the name "linear-law" attrition process.

The time solution for equations (8) and (9) is given by (see G. Weiss [Ref. 35])

$$x(t) = \frac{(y_o-b/ax_o)x_o \exp((b/ax_o-y_o)at)}{(y_o-b/ax_o \exp((b/ax_o-y_o)at)} \quad (13)$$

$$y(t) = y_o-b/ax_o + b/ax(t) \quad (14)$$

Others have suggested general forms of homogeneous force models given by

$$\frac{dx}{dt} = - F(t,x,y) \quad (15)$$

$$\frac{dy}{dt} = - G(t,x,y) \quad (16)$$

For example, Willard [Ref. 37] considered the following Lanchester-type equations:

$$\frac{dx}{dt} = - ax^c f(t,x,y) \quad (17)$$

$$\frac{dy}{dt} = - by^c f(t,x,y) \quad (18)$$

Many "general" forms contain variable coefficients (e.g., $a = a(t)$ and $b = b(t)$). Taylor [Ref. 30] has shown that except for some very special cases, the solutions to variable coefficient Lanchester-type equations are very complex and in many cases of little practical use.

B. STOCHASTIC MODELS

Koopman [Ref. 22] suggested a reformulation of the attrition process in stochastic form. The resulting stochastic attrition process has been appropriately called by him the Lanchester stochastic process. Others have subsequently employed a stochastic analysis of combat. With the inclusion of random variations in the attrition process, analysts have attempted to better represent the complex random process of combat with the goal of gaining insights into combat not available from deterministic models.

Snow [Ref. 27] showed that the following assumptions yield the square-law attrition process:

(A1) $P(t+\Delta t,m,n:t,m+1,n) = an\Delta t$

(A2) $P(t+\Delta t,m,n:t,m,n+1) = bm\Delta t$

(A3) Probability of more than one casualty occurring in a time of length Δt is of the order of magnitude $o(h)$ where
$$\lim_{h \to 0} \frac{o(h)}{h} = 0.$$

From these, he developed the Chapman-Kolmogorov forward equations:

$$\frac{dP(t,m,n)}{dt} = anP(t,m+1,n) + bmP(t,m,n+1) - (an+bm)P(t,m,n) \tag{19}$$

If one assumes, instead

(A1) $P(t+\Delta t,m,n:t,m+1,n) = F(t,m+1,n)\Delta t$

(A2) $P(t+\Delta t,m,n:t,m,n+1) = G(t,m,n+1)\Delta t$

(A3) Probability of more than one casualty occurring in a time of length Δt is of magnitude $o(h)$.

then the resulting Chapman-Kolmogorov forward equations are:

$$\frac{dP(t,m,n)}{dt} = F(t,m+1,n)P(t,m+1,n) + G(t,m,n+1)P(t,m,n+1)$$
$$- (F(t,m,n) + G(t,m,n))P(t,m,n) \qquad (20)$$

In this thesis, this process will be called the <u>general Lanchester-type stochastic attrition process</u>.

Work has been done by several authors to obtain general solutions for specific stochastic attrition processes. Brown [Ref. 7] attempted to find the time state probabilities ($P(t,m,n)$) for a class of homogeneous force stationary Markov attritions processes (that is, when $F(t,m,n) = A(m,n)$ and $G(t,m,n) = B(m,n)$). He concluded that the expression derived is too complicated for practical use. Isbell and Marlow [Ref. 14] developed a general solution to (20) for a square-law stochastic attrition process. If $F(t,m,n) = an + \beta m$ and $G(t,m,n) = bm + \alpha n$ (Isbell and Marlow referred to α and β as operational loss rates), with $a + \alpha = b + \beta$, then

$$P(t,m,n) = F(m,n:m_o,n_o)\binom{m_o+n_o}{m+n}(\exp((b+\beta t)-1)^{m_o+n_o-m-n}$$
$$\cdot \exp(-(b+\beta)(m_o+n_o)t) \qquad (21)$$

where $F(m,n:m_o,n_o)$ is developed from a recursive relationship

$$F(m,n:m_o,n_o) = \frac{m_o+an_o}{(b+\beta)m_o+(a+\alpha)n_o} F(m,n:m_o-1,n_o)$$

$$+ \frac{n_o+bm_o}{(b+\beta)m_o+(a+\alpha)n_o} F(m,n:m_o,n_o-1) \quad (22)$$

where $F(m_o,n_o:m_o,n_o) = 1$.

Clark [Ref. 8] developed a general solution for the time state probabilities for the linear-law stochastic attrition process (i.e., $F(t,m,n) = amn$ and $G(t,m,n) = bmn$);

$$P(t,m,n) = \sum_{j=m}^{m_o} \sum_{k=n}^{n_o} \left\{ \frac{(-1)^{k-n+j-m} a^{m_o-m} b^{n_o-n} (m_o)! (n_o)!}{(a+b)^{n_o-n+m_o-m} m! n! (k-n)! (j-m)! (n_o-k)!} \cdot \left[\prod_{\ell=1}^{j-m} \frac{(j-n-\ell)}{(j-k-\ell)} \prod_{\ell=1}^{m_o-j} \frac{(n_o-j+\ell)}{(j+k+\ell)} \right] \exp(-(a+b)jkt \right\} \quad (23)$$

(However, he did not compare his results with those for the probability of winning (i.e., "true" absorption probabilities). This expression is too complicated to provide any insights by itself. This author has no knowledge of any other general solutions developed for the time state probabilities. It is, however, a straight forward task to compute $P(t,m_o,n)$ and $P(t,m,n_o)$, which are used in the included computer program as a partial check for accuracy.

Work has also been done to develop exact expressions for the probability of winning, for a stationary Markov attrition process. The exact expression for the probability of winning for the linear-law stochastic attrition process is easily obtained from classical random walk results [Ref. 28]:

$$P_Y = \sum_{n=1}^{n_o} P(o,n:m_o,n_o) = \sum_{n=1}^{n_o} \binom{m_o+n_o-n-1}{m_o-1} \left(\frac{a}{a+b}\right)^{m_o} \left(\frac{b}{a+b}\right)^{n_o-n} \tag{24}$$

Smith [Ref. 26] extended work done by Brown [Ref. 7] and developed the explicit expression for the square-law stochastic attrition process:

$$P_X = \sum_{m=1}^{m_o} P(m,0:m_o,n_o) \tag{25}$$

where

$$P(m,0:m_o,n_o) = \left[\frac{bm_o}{an_o+bm_o}\right] \sum_{j=m}^{m_o} \left[\frac{(-1)^{m_o-j}\, j^{m_o+n_o-m-1}}{\Gamma(m_o-j+1)\;\Gamma(n_o+b/a\;j+1)} \frac{\Gamma(b/a\;j+1)}{(j-m+1)}\right] \tag{26}$$

This is an impressive looking expression, but from a practical viewpoint, it is not very useful. A computer is needed to solve this equation for any appreciable force sizes, but one immediately runs into overflow and underflow problems. To solve these complications, recursive relationships must be used; but if one looks carefully at the relationships used, he is lead back to the initial recursive relationships used to obtain the solution:

$$P(0,n:m_o,n_o) = \frac{A(m_o,n_o)}{A(m_o,n_o)+B(m_o,n_o)} P(0,n:m_o-1,n_o)$$

$$+ \frac{B(m_o,n_o)}{A(m_o,n_o)+B(m_o,n_o)} P(0,n:m_o,n_o-1) \qquad (27)$$

It was found by this author that the most efficient method to solve the stochastic model for the probability of winning was by using the above recursive relationship, or one developed by Springall [Ref. 28], not the general force level solutions (See Appendix A).

Springall developed a recursive relationship for the distribution of the time of battle termination:

$$E(t^r) = \sum_{n=n_{bp}+1}^{n_o} A(m_{bp}+1,n) X^{(r)}_{m_{bp}+1}$$

$$+ \sum_{m=m_{bp}+1}^{m_o} B(m,n_{bp}+1) X^{(r)}_{m,n_{bp}+1} \qquad (28)$$

where

$$X^{(0)}_{m_o,n_o} = \frac{1}{A(m_o,n_o)+B(m_o,n_o)} \qquad (29)$$

$$X^{(0)}_{m_o,n} = \frac{B(m_o,n+1) X^{(0)}_{m_o,n+1}}{A(m_o,n)+B(m_o,n)} \qquad (30)$$

$$X^{(0)}_{m,n_o} = \frac{A(m+1,n_o) X^{(0)}_{m+1,n_o}}{A(m,n_o)+B(m,n_o)} \qquad (31)$$

$$X_{m,n}^{(0)} = \frac{A(m+1,n)X_{m+1,n}^{(0)}+B(m,n+1)X_{m,n+1}^{(0)}}{A(m,n)+B(m,n)} \tag{32}$$

for $i \geq 1$

$$X_{m_o,n_o}^{(i)} = \frac{iX_{m_o,n_o}^{(i-1)}}{A(m_o,n_o)+B(m_o,n_o)} \tag{33}$$

$$X_{m_o,n}^{(i)} = \frac{iX_{m_o,n}^{(i-1)}+B(m_o,n+1)X_{m_o,n+1}^{(i)}}{A(m_o,n)+B(m_o,n)} \tag{34}$$

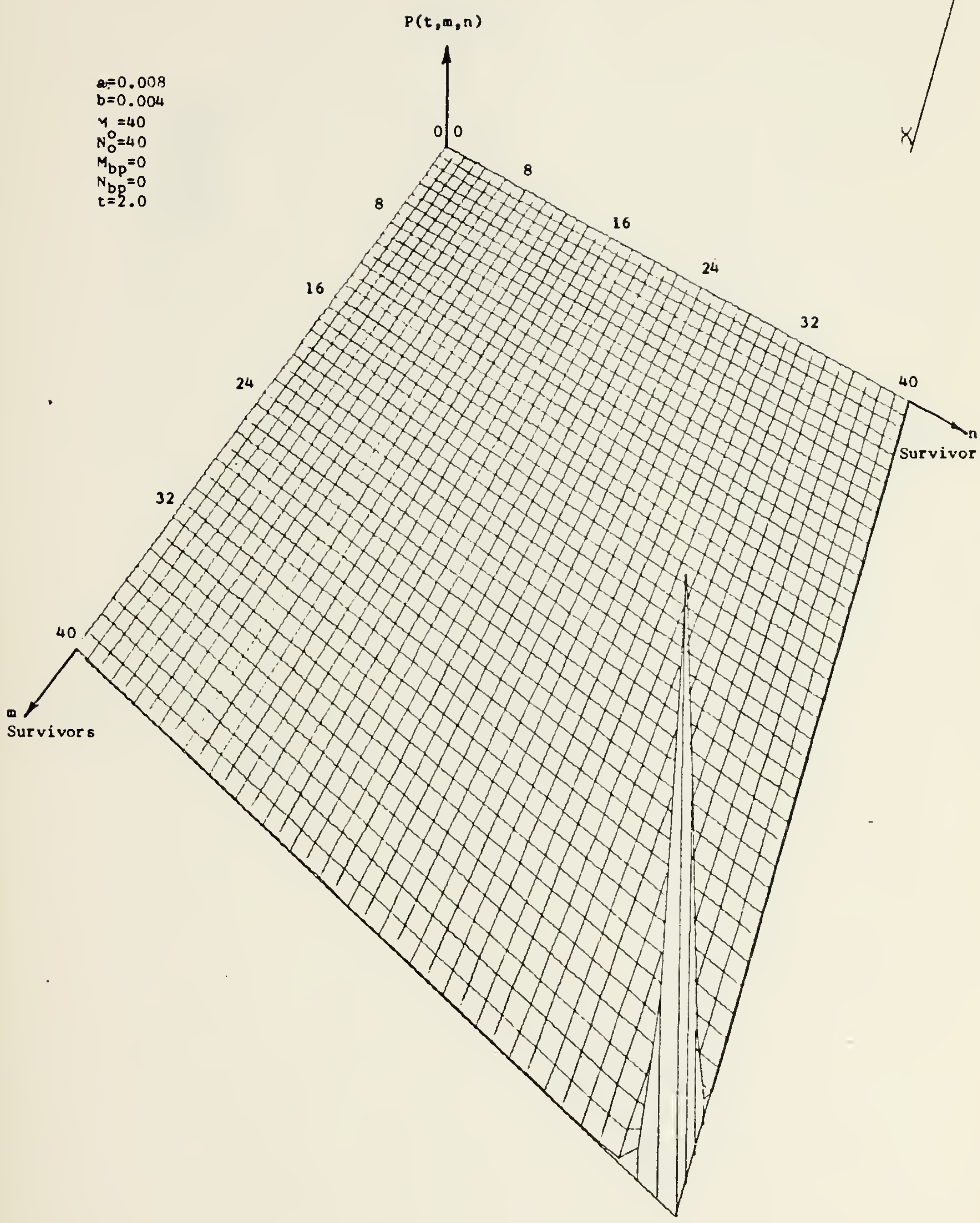
$$X_{m,n_o}^{(i)} = \frac{iX_{m,n_o}^{(i-1)}+A(m+1,n_o)X_{m+1,n_o}^{(i)}}{A(m,n_o)+B(m,n_o)} \tag{35}$$

$$X_{m,n}^{(i)} = \frac{iX_{m,n}^{(i-1)}+A(m+1,n)X_{m+1,n}^{(i)}+B(m,n+1)X_{m,n+1}^{(i)}}{A(m,n)+B(m,n)} \tag{36}$$

From studying the equations presented in this section, one gains an appreciation for the increase in complexity of the model by the inclusion of uncertainty in the attrition process. It is apparent that, although exact solutions may be obtained for some stochastic models, numerical results are not readily generated by hand. A computer is therefore essential for the numerical solution of the simple stochastic models.

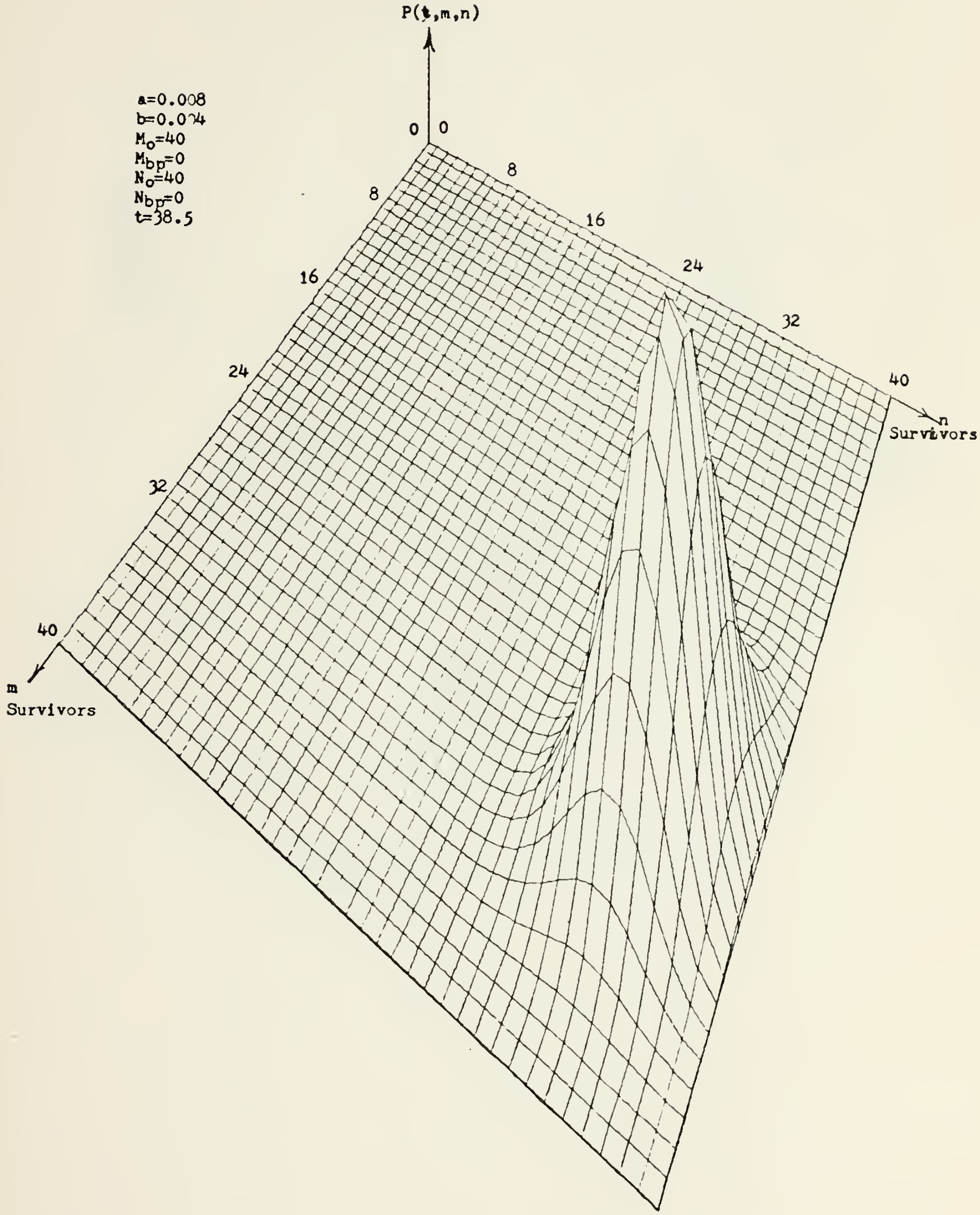
It is surprising that more use has not been made of the computer in investigating stochastic models, especially the

use of computer graphics. For example, the five following figures are plots of the time state probabilities for a stochastic model at 3, 25, 50, 75, and 100% of the time to completion of the equivalent deterministic model. It is believed that something of this nature has not been utilized in the open literature before. It gives some interesting insights into the progress of a stochastic model. The state probability seems to start as a spike at (m_o, n_o) and progresses over time somewhat as a drop of water would. As time increases, the drop spreads and moves toward the banks (boundaries of the state space). The angle at which the wave moves is determined by the relationship between the attrition rates of the process (in this case, the model is the square-law stochastic attrition process, so the relationship is between a, b, m, and n). As the "wave" hits the wall, it "sticks" (is absorbed), thus giving, as $t \to \infty$, the distribution of win probabilities. (It is interesting to note that this relates the work of Clark [Ref. 8], Weiss [Ref. 35], Morse and Kimball [Ref. 22], and Snow [Ref. 27], who were concerned with the solution to the Chapman-Kolmogorov, to the work of Smith [Ref. 26], Brown [Ref. 7], etc., who were concerned with the win probabilities). Note that in the present case, Y has a very large probability of winning. If the forces were moved closer to parity, more of the wave would be absorbed by the Y force boundary, giving X a larger probability of winning.



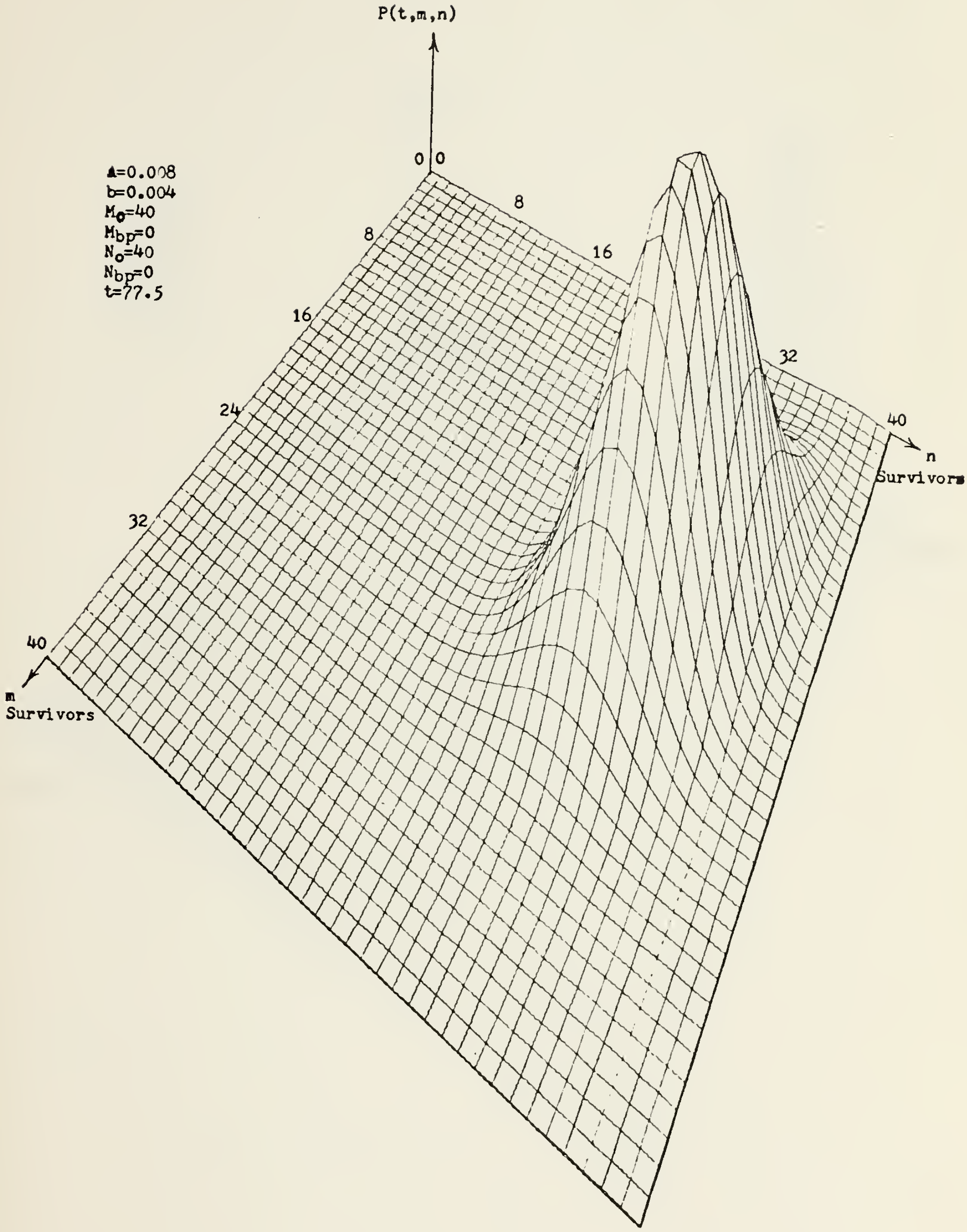
Plot of P(t,m,n) for Fixed t

FIGURE 1



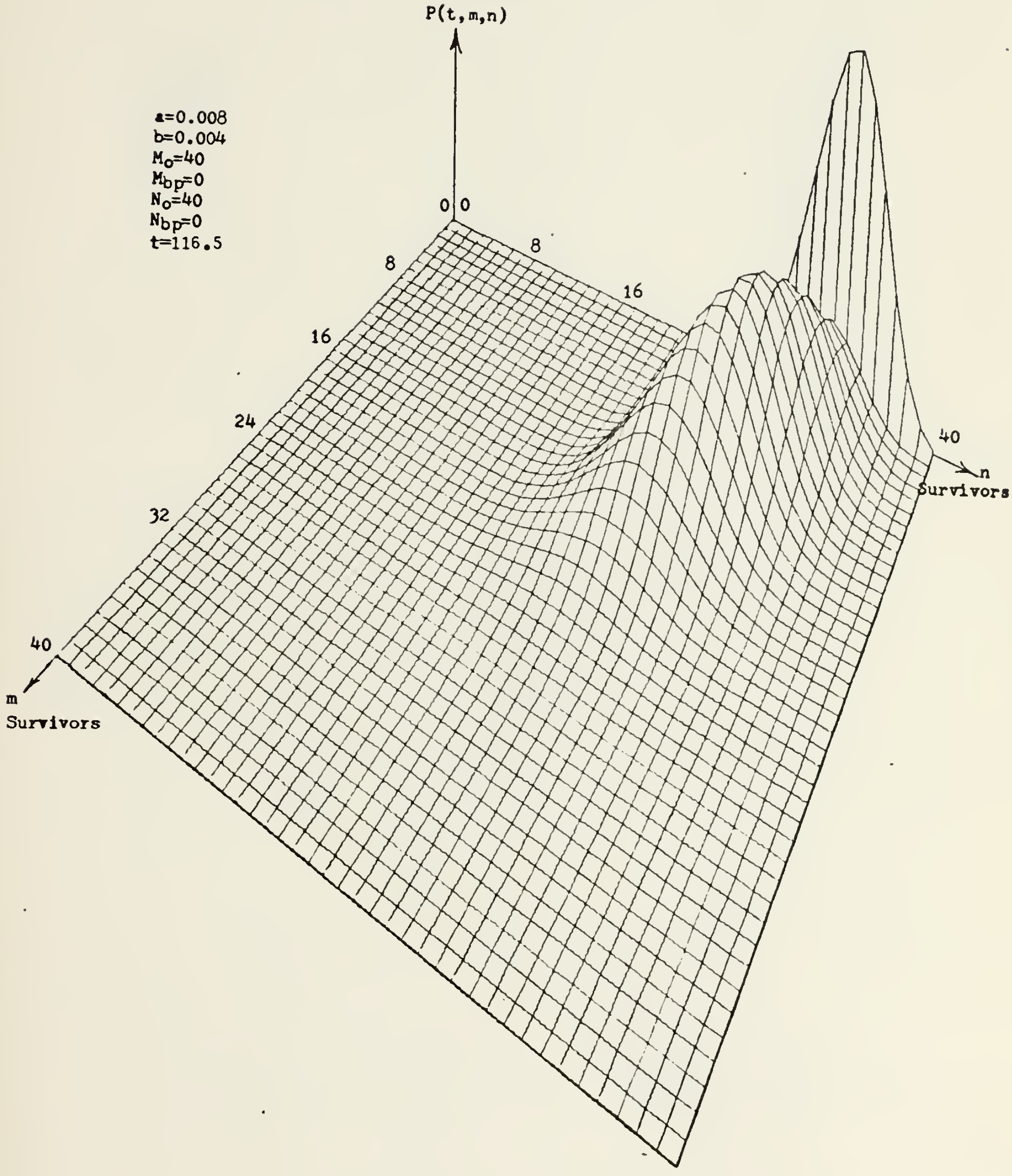
Plot of P(t,m,n) for Fixed t

FIGURE 2



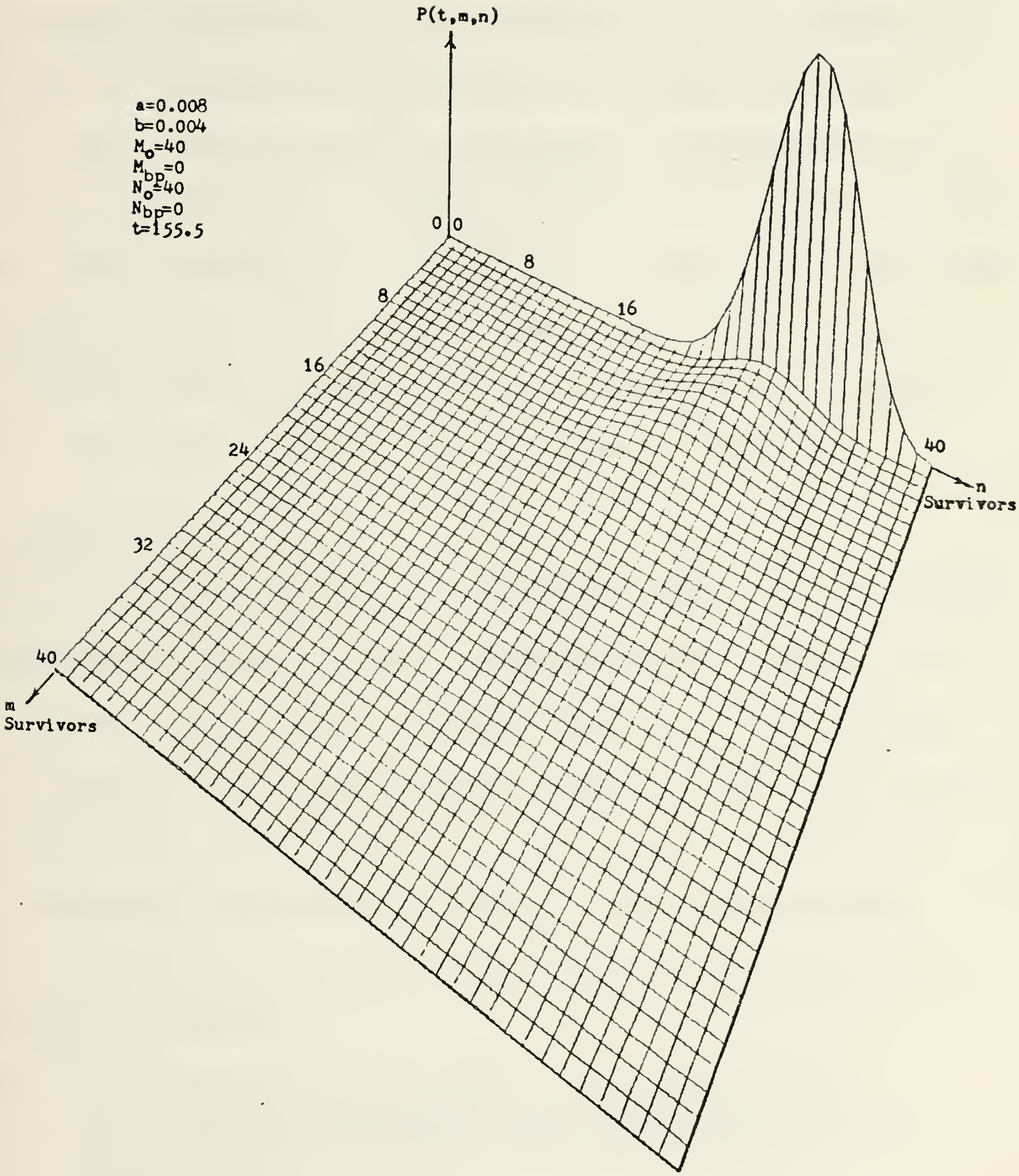
Plot of P(t,m,n) for Fixed t

FIGURE 3



Plot of P(t,m,n) for Fixed t

FIGURE 4



Plot of P(t,m,n) for Fixed t

FIGURE 5

The above discussion has given a description of what occurs with the probability mass over time. Another description of the process is, it acts as a combination of diffusion and convective transport of probability mass. This concept enhances ones understanding of the nature of so-called diffusion approximations of Markov processes (i.e., state space assumed continuous). Additionally, it provides insights into temporal changes in variance in force levels (see Chapter III, Section C). Thus, it is felt that computer graphics, if utilized more fully could provide easily obtainable insights into the dynamics of a combat model, that would be difficult to obtain otherwise.

C. COMPARISONS

Two types of comparisons have been made between stochastic and deterministic models; probability of winning, and the expected force levels against the time history of the force levels of the deterministic model. Two recent studies have been done comparing the probability of winning with the certainty of winning or losing in the deterministic model [Ref. 20 and Ref. 28]. Both came up with basically the same conclusions.

(1) As force levels increase, the probability of winning approaches the certain probability of winning or losing in the deterministic model.

(2) Significant differences occur only when the forces are near parity.

Lee and Wannasilpa [Ref. 20] came to their conclusions for the square-law, linear-law, and mixed-law. Springall arrived at the same conclusions for a slightly more complicated model.

Snow [Ref. 27] showed that the time history of the expected force levels differed from the time history of the deterministic force levels; that is $\overline{m}(t)-x(t) \neq 0$. Clark [Ref. 8] defined this as bias. This seems to be an appropriate term and will be used here. Snow apparently first developed an explicit expression for this bias for the square-law;

$$\frac{d\overline{m}}{dt} = - a\overline{n} + a \sum_{n=1}^{n_o} P(t,0,n) \tag{37}$$

Clark [Ref. 8] empirically showed that bias exists for the linear-law. All researchers seem to have arrived at the same general conclusions:

(1) As initial force levels increase, the bias decreases.

(2) The longer a battle lasts, the greater the bias.

Powers and Taylor [Ref. 24] and Hanna [Ref. 12] have compared differences in optimal time sequential fire distribution policies for deterministic and stochastic models: Hanna's investigation indicates that there is a real difference between the two (at least for small numbers of total combatants).

Thus, some work has been done to investigate the differences between deterministic and stochastic models, especially in the areas of who is going to win and the average time histories of the force levels. However, the analysis done up to the present has been somewhat limited in scope and does not allow an analyst to say with any degree of certainty, when the complex random process of combat can be adequately represented by a deterministic model.

III. COMPARISON OF DETERMINISTIC AND STOCHASTIC MODELS FOR "SQUARE LAW" ATTRITION PROCESS

It would be most useful to be able to make comparisons between stochastic and deterministic models used today. This is not feasible within the scope of this paper for several reasons.

1. It is not clear, in many cases, when the two models are equivalent. For example, one could try and compare DYNTACS-X and Bonder/I.U.A., but before this comparison could be made, one would have to insure that the input parameters are equivalent. In the case of these two models, that is not a trivial task, as the structure of the two models is quite different, particularly in route selection. About the best that can be done is to construct the input parameters for each model and assume they are equivalent.
2. Parametric analysis is not feasible in a complicated model. Complicated models, both stochastic and deterministic, are expensive to run. Because of this and the complex interactions among the variables of a model, parametric analysis is not only prohibitively expensive, but difficult to do.
3. It is difficult to gain insights in a comparison of simple stochastic and deterministic models, and

essentially impossible to gain many useful insights in a comparison of complicated models.

Lanchester-type models may be thought of as resulting from an aggregation of the myriad of combat environmental variables into constant attrition rate coefficients and time varying force levels and, moreover, as complementary complex system models. Such idealizations will be considered in this thesis with the knowledge they are idealizations, but that they may provide insights that can be used as points of departure for the investigation of more complex models of combat. Specifically, two models will be compared; the deterministic Lanchester square-law and the square-law stochastic attrition process. It is believed that trends found from this investigation may be carefully used to determine trends for a comparison of more complex models.

As was shown in the literature survey, the square-law is a special case of the general differential equations

$$\frac{dx}{dt} = - F(t,x,y) \tag{15}$$

$$\frac{dy}{dt} = - G(t,x,y) \tag{16}$$

where

$$F(t,x,y) = ay$$

and

$$G(t,x,y) = bx$$

giving

$$\frac{dx}{dt} = -ay \tag{1}$$

$$\frac{dy}{dt} = -bx \tag{2}$$

with initial conditions

$$x(t = 0) = x_o \tag{3}$$

$$y(t = 0) = y_o \tag{4}$$

Also shown was the general form of the stochastic model:

$$\frac{dP(t,m,n)}{dt} = F(t,m+1,n)P(t,m+1,n) + G(t,m,n+1)P(t,m,n+1)$$

$$- (F(t,m,n) + G(t,m,n))P(t,m,n) \tag{20}$$

with boundary conditions

$$\frac{dP(t,m_{bp},n_{bp})}{dt} = F(t,m_{bp}+1,n_{bp})P(t,m_{bp}+1,n_{bp})$$

$$+ G(t,m_{bp},n_{bp}+1)P(t,m_{bp},n_{bp}+1) \tag{38}$$

$$\frac{dP(t,m,n_{bp})}{dt} = F(t,m+1,n_{bp})P(t,m+1,n_{bp})$$

$$+ G(t,m,n_{bp}+1)P(t,m,n_{bp}+1)$$

$$- F(t,m,n_{bp})P(t,m,n_{bp}) \qquad (39)$$

$$m > m_{np}$$

$$\frac{dP(t,m_{np},n)}{dt} = F(t,m_{np}+1,n)P(t,m_{np}+1,n)$$

$$+ G(t,m_{np},n+1)P(t,m_{np},n+1)$$

$$- G(t,m_{np},n)P(t,m_{np},n) \qquad (40)$$

$$n > n_{bp}$$

and initial conditions

$$P(0,m,n) = \begin{cases} 1 & m = m_o \text{ and } n = n_o \\ 0 & \text{otherwise} \end{cases} \qquad (41)$$

Substituting the square-law attrition rates gives the square-law stochastic attrition process:

$$\frac{dP(t,m,n)}{dt} = anP(t,m+1,n) + bmP(t,m,n+1) - (an+bm)P(t,m,n) \qquad (19)$$

and boundary conditions

$$\frac{dP(t,m_{np},n_{bp})}{dt} = an_{bp}P(t,m_{bp}+1,n_{bp}) + bm_{bp}P(t,m_{bp},n_{bp}+1) \tag{41}$$

$$\frac{dP(t,m,n_{bp})}{dt} = an_{bp}P(t,m+1,n_{bp}) + bmP(t,m,n_{bp}+1) - an_{bp}P(t,m,n_{bp}) \tag{42}$$

$$\frac{dP(t,m_{np},n)}{dt} = anP(t,m_{np}+1,n) + bm_{bp}P(t,m_{bp},n+1) - bm_{bp}P(t,m_{np},n) \tag{43}$$

and initial conditions

$$P(o,m,n) = \begin{cases} 1 & m = m_o \text{ and } n = n_o \\ 0 & \text{otherwise} \end{cases} \tag{44}$$

It was noted that the square-law comparisons have been made previously. However, as far as the author can tell, no previous comparisons have been made contrasting the models for breakpoint rules other than annihilation. Taylor [Ref. 30] has pointed out (and Adkins [Ref. 1] examined in further detail) that a neglected area in the modeling of land combat is the modeling of battle termination. In actual battles, several different events might cause battle termination. If one of the opposing forces is annihilated, then the battle ends; however, this is a rare event essentially never observed

in real combat, with a few exceptions (e.g., Iwo Jima, Alamo, etc.). If one of the opposing forces surrenders unconditionally, the battle will also terminate. The third and most common event which will result in battle termination is that of one of the opposing force breaking contact with the enemy and withdrawing from the battlefield [Ref. 13]. There are other possible events which might result in battle termination, but they will not be considered.

For the purposes of further discussion, a <u>breakpoint</u> is defined to be that state of a battle at which a unit considers itself no longer capable of performing its mission and as a result elects to break contact with the enemy and withdraw from the battlefield. Therefore, when a unit withdraws from the battlefield strictly to avoid further combat, the unit is considered to have reached its breakpoint and lost the engagement.

In many current land combat models a unit's breakpoint is determined by the percentage of casualties sustained by that unit [Ref. 1]. Although available empirical evidence shows that a deterministic battle termination model does not appear reasonable, it is an adequate "first cut" [Ref. 13]. And a breakpoint determined by the percent of casualties sustained is more reasonable than one that assumes the battle goes to annihilation. For this reason, the investigation was done with deterministic breakpoints determined by the percent of casualties.

A. PROBABILITY OF WINNING

Hypothesis 1-1

> As the breakpoint force levels, m_{bp} and n_{bp} are moved closer to the initial force levels, m_o and n_o, respectively; the difference in the probability of winning between the two models increases.

Lee and Wannasilpa [Ref. 20] deduced that as the initial force levels are increased, the difference between the probability of winning in the deterministic model and the stochastic model decreases. In fact it appeared that the difference disappears in the limit. However, the conclusion reached by Lee and Wannasilpa was based only on battles to annihilation. A look at the recursive relationship used in Appendix A to solve for the probability of winning is based on a summation from $m_{bp}+1$ to m_o. Furthermore, the recursive relationships are from m_{bp} to m_o. These seem to indicate that the differences in the probability of winning decrease not just as m_o increases, but as m_o-m_{bp} increases. The following figure seems to support this. Note, that for a deterministic model, the probability that Y wins, P_Y, has a discontinuous jump from one to zero as the ratio $\frac{bm_o^2(1-(1-F_X)^2)}{an_o^2(1-(1-F_Y)^2)}$ passes through one. (Note: F_X and F_Y are the fractions of initial combatants each side is willing to lose.) For the square-law stochastic attrition process, however, P_Y moves continuously from one to zero reaching about 0.5 at a ratio of one. Note that as F_X and F_Y decrease, m_{bp} and n_{bp} increase, so m_o-m_{bp} and

P_Y for Varying Breakpoints

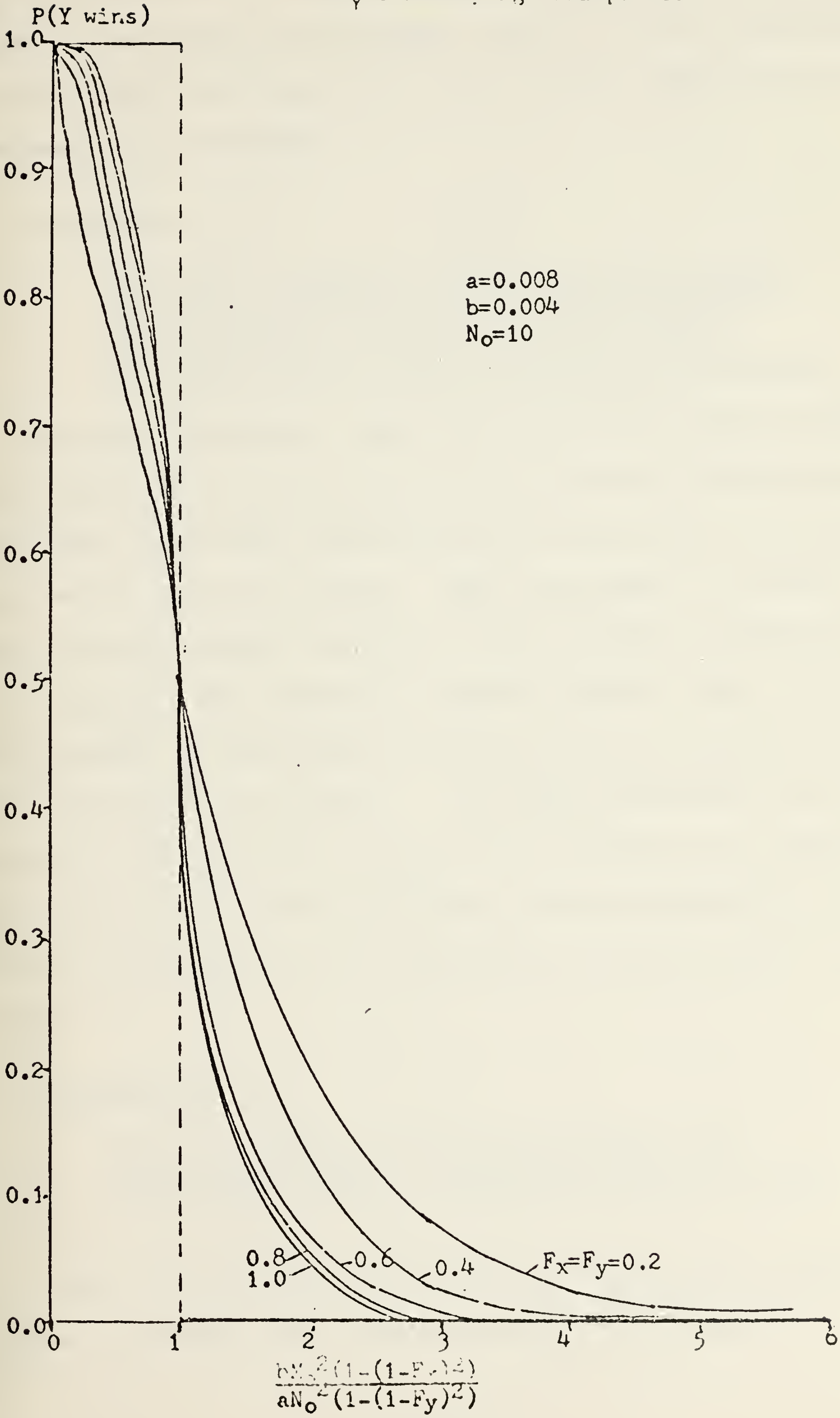


FIGURE 6

$n_o - n_{bp}$ decrease. As they decrease, P_Y for the stochastic model moves further and further from P_Y for the deterministic model. This was found to be true for all combinations of parameters considered.

Hypothesis 1-2

> For fixed breakpoint casualty/initial force ratios (F_X and F_Y), the difference between the probability of winning for the deterministic model and the stochastic model decreases as the initial force levels increase.

Lee and Wannasilpa [Ref. 20] deduced the above except their conclusion was restricted to battles to annihilation. The above hypothesis generalizes the conclusion to include any set of fixed F_X and F_Y. Once hypothesis 1-1 along with Lee and Wannasilpa's conclusion are accepted, hypothesis 1-2 follows. The following figures support this. It is interesting to note that the plots of P_Y are similar for all fixed sets of F_X and F_Y. The only difference is in the angles of the curves. As F_X and F_Y increase, the result is similar to rotating the curves counter-clockwise. This causes an increase in the difference between P_Y for the two models.

Hypothesis 1-3

> As the forces move away from parity, the difference between the probability of winning for the two models becomes negligible.

This was the final conclusion reached by Lee and Wannasilpa. Although their conclusion was for battles to

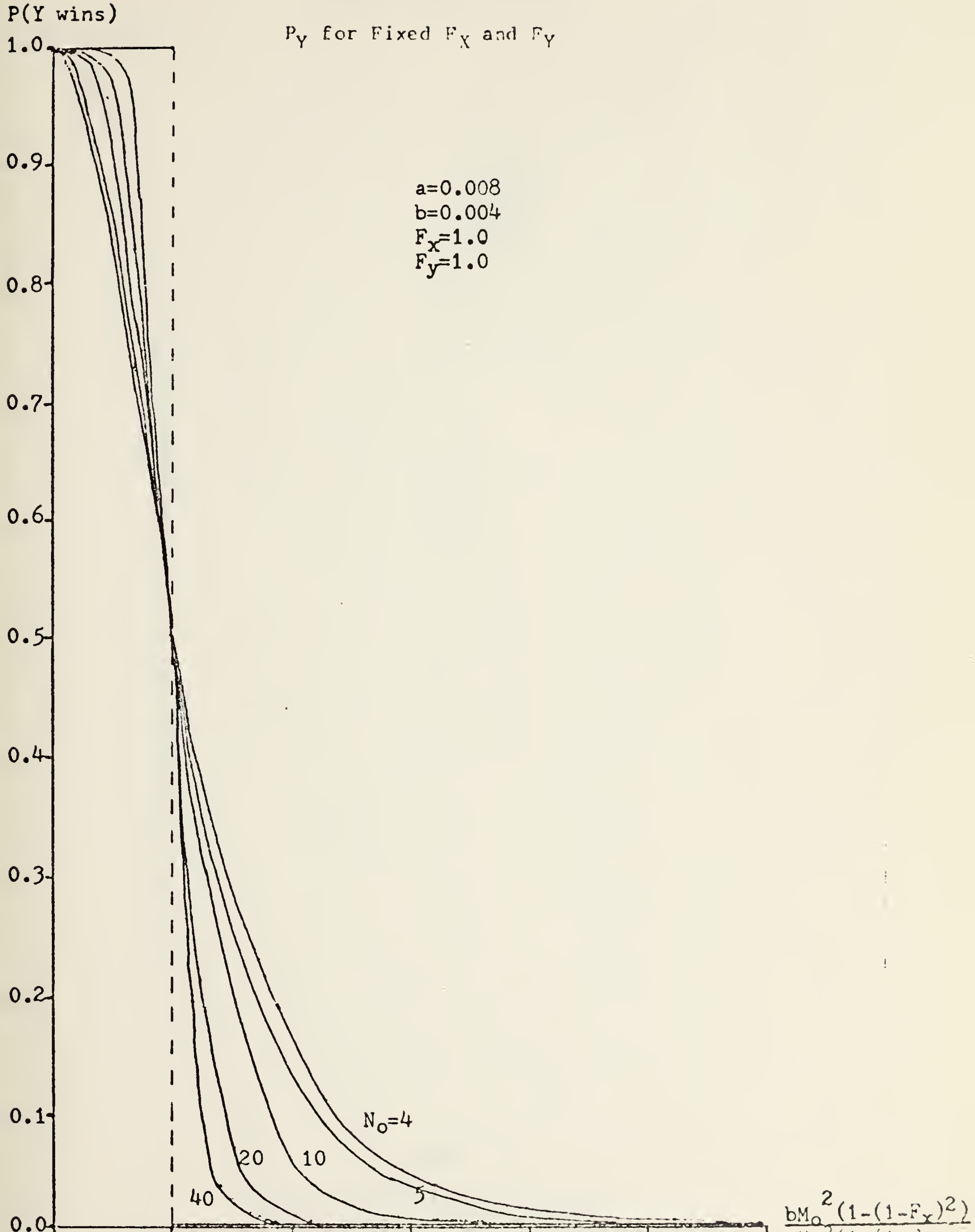

P(Y wins)
P_Y for Fixed F_X and F_Y
a=0.008
b=0.004
F_x=1.0
F_y=1.0
N_o=4
5
10
20
40
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
0
1
2
3
4
5
6
$\frac{bM_o^2(1-(1-F_x)^2)}{aN_o^2(1-(1-F_y)^2)}$

FIGURE 7

P_Y for Fixed F_X and F_Y

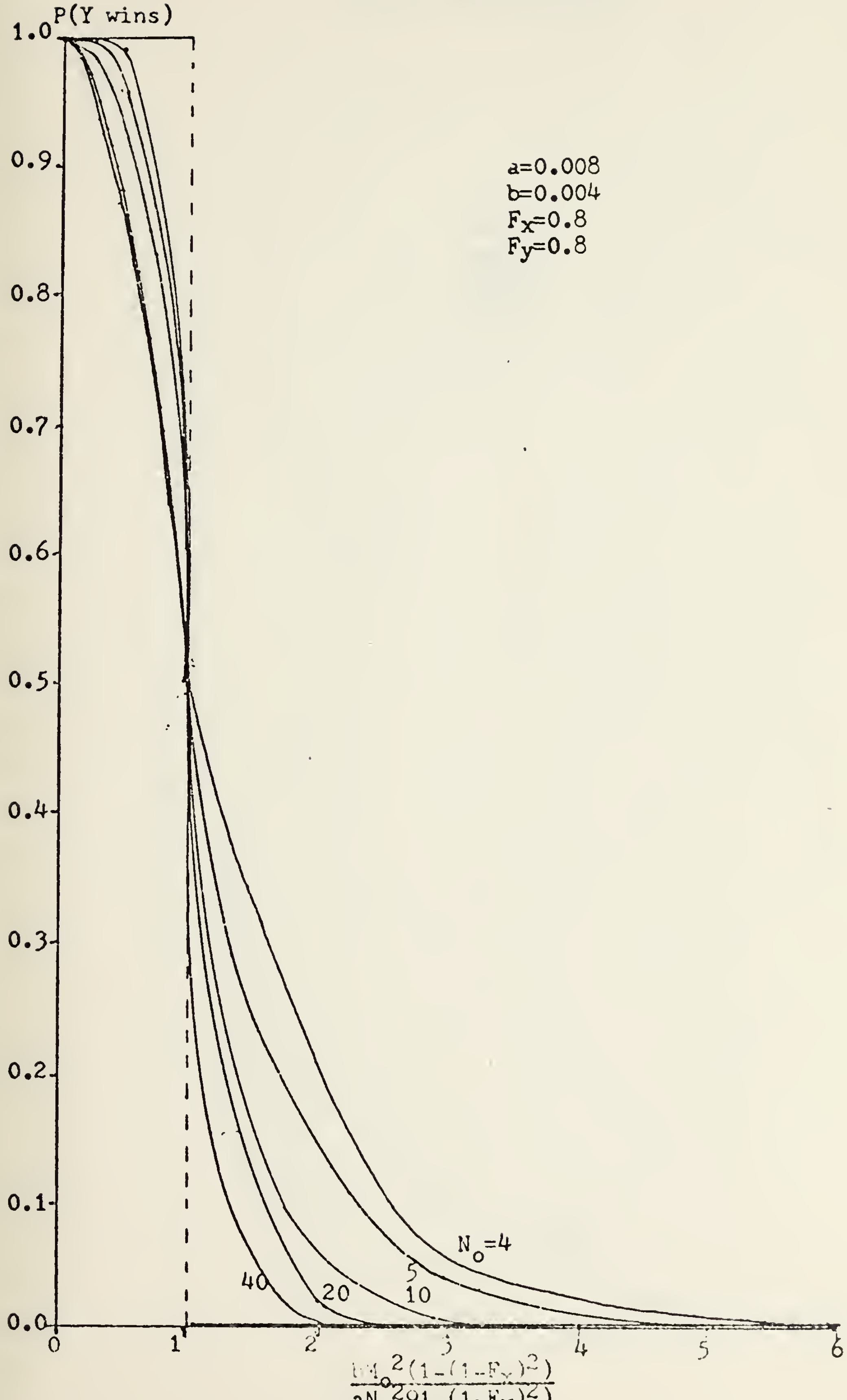
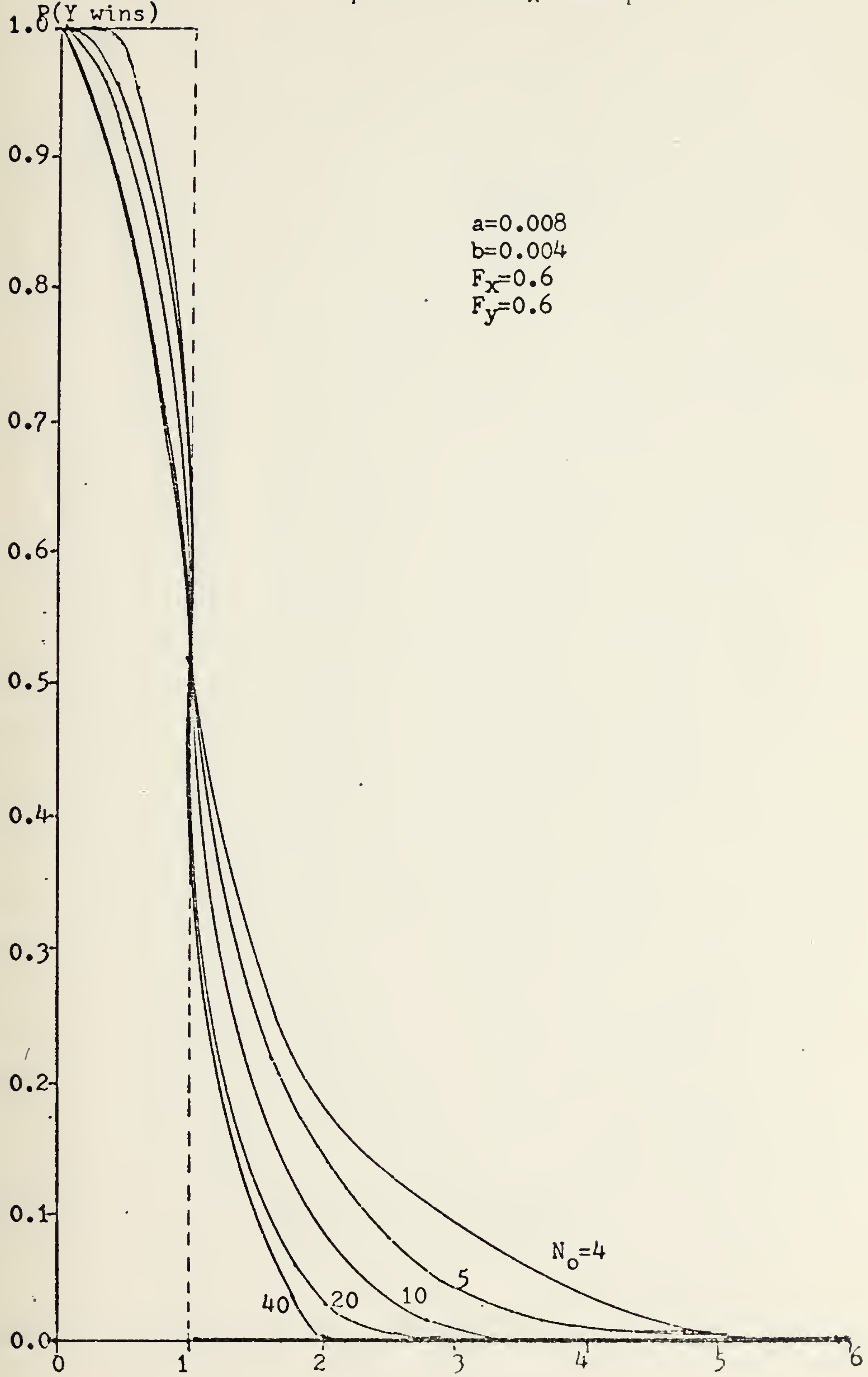


FIGURE 8

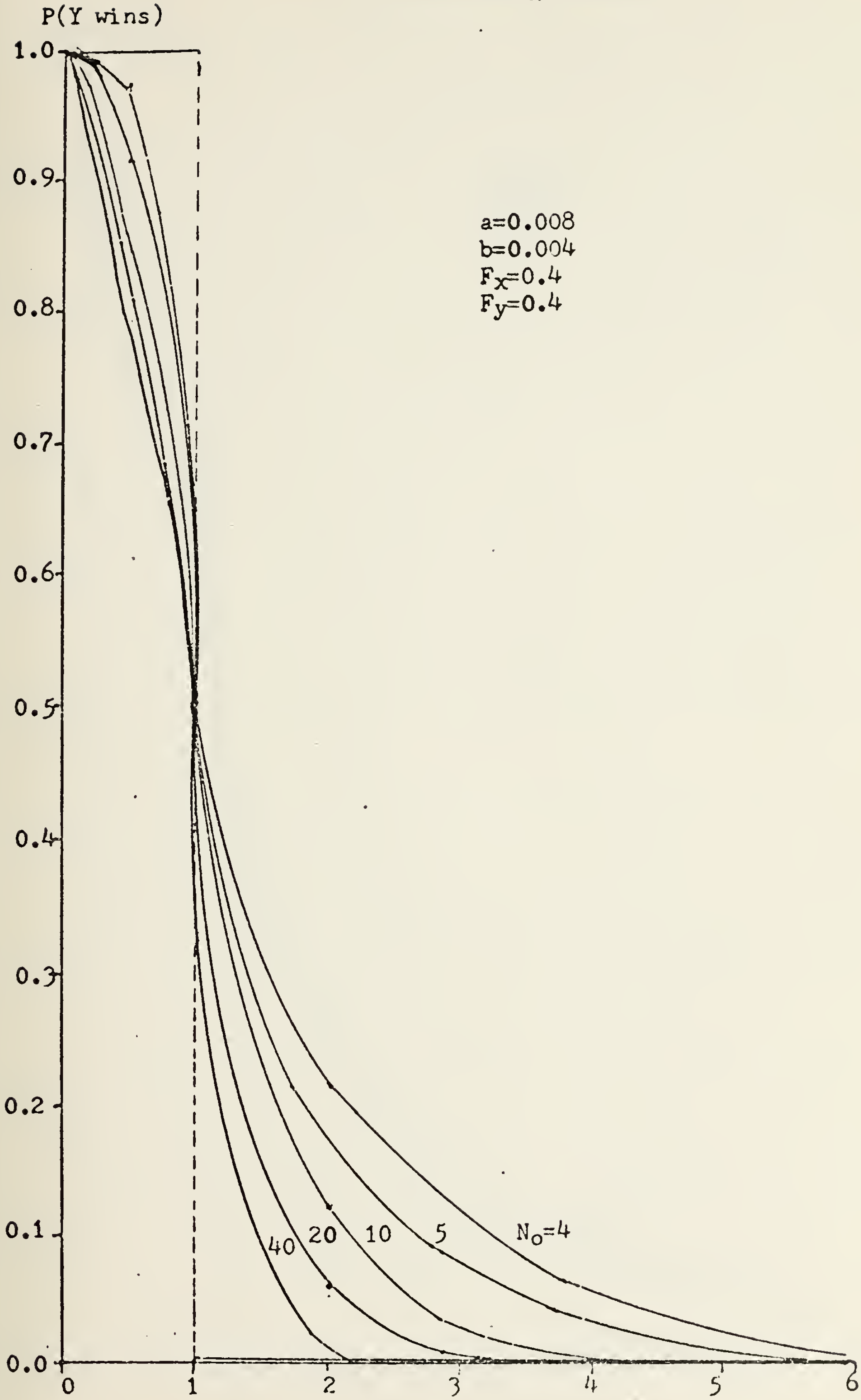
P_Y for Fixed F_X and F_Y



$$\frac{bM_o^2(1-(1-F_x)^2)}{aN_o^2(1-(1-F_y)^2)}$$

FIGURE 9

P_Y for Fixed F_X and F_Y



$$\frac{bM_o^2(1-(1-F_x)^2)}{aN_o^2(1-(1-F_y)^2)}$$

FIGURE 10

P_Y for Fixed F_X and F_Y

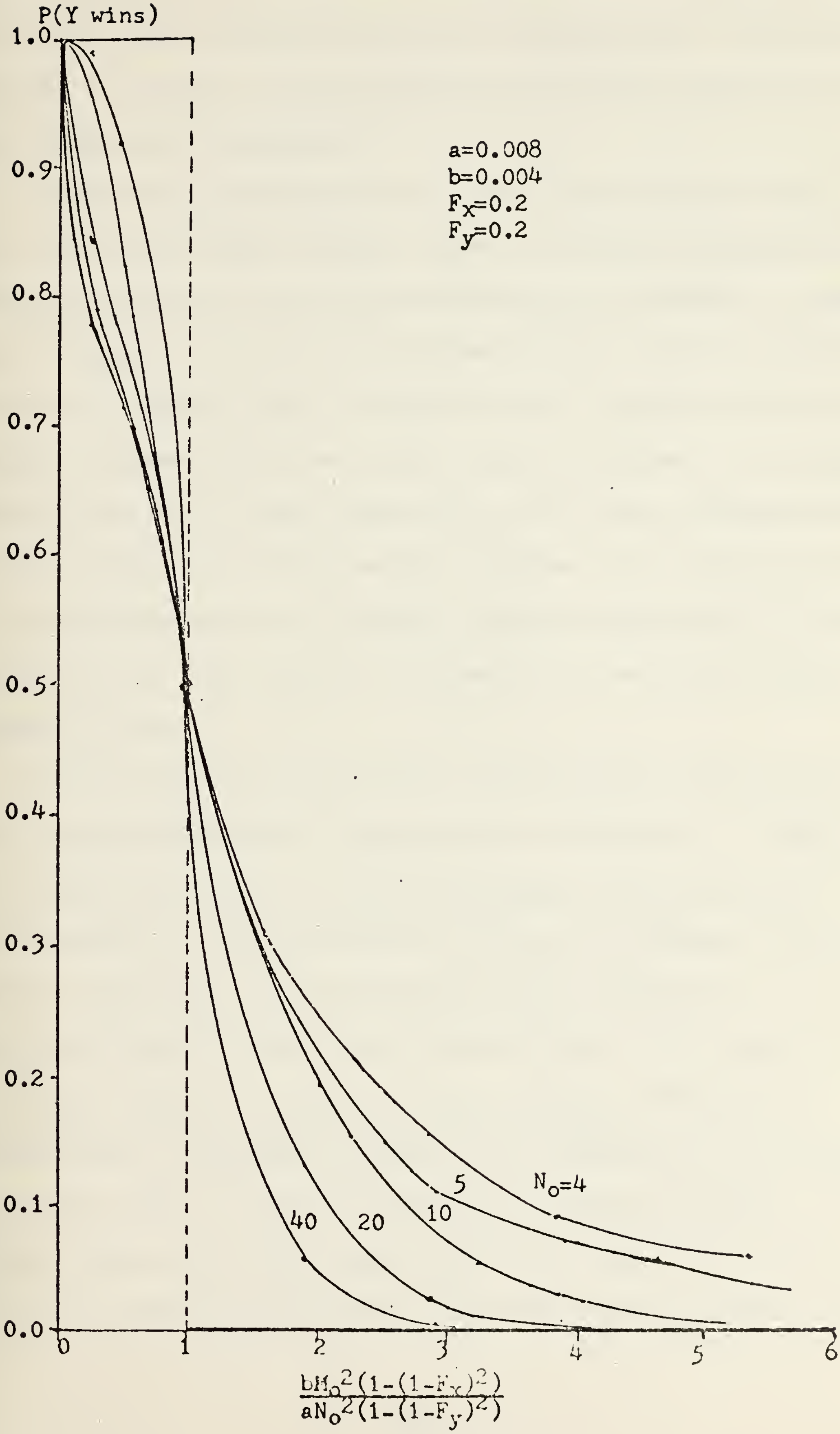


FIGURE 11

annihilation, the present findings do not alter the conclusion for other fixed force level breakpoints. The only change is in how far from parity the forces must be to make the difference negligible.

The general consequence of this hypothesis is that unless the forces are near parity, there is little difference in the probability of winning between the two models. From this viewpoint, then, if one side is going to win decisively, there is little reason to use the more expensive stochastic model. Indeed, this has some intuitive appeal: The more decisively a side wins the battle, the less influence any uncertainties will have on the outcome. On the other hand, if the forces are near parity, the uncertainties in the evolution of the dynamics of combat may cause the expected loser to win.

If the previous hypotheses are combined with this hypothesis, some interesting trends can be observed. It has been concluded that, with increasing $m_o - m_{bp}$ and $n_o - n_{bp}$, the differences in the probability of winning between the two models decreases. When combined with hypothesis 1-3, this says that, with casualties on each side on the order of 20 or more, and with forces not near parity, there is no significant difference between the probability of winning for each model. Even if the forces are near parity, if the force levels are very large, and both sides are willing to take 20 or more casualties; the difference in model win

probabilities is negligible. In these cases, if only the win probabilities were considered as measurement criteria between the two models, one would use the less expensive deterministic model as an acceptable substitute for the more realistic stochastic model.

One must be cautious of this conclusion. It would be inaccurate to make this conclusion unless the only objective of the model is to determine which side will win. In most cases, there are other insights analysts try to gain from the model, such as how many casualties does each side take, at what points in the battle are the most casualties being taken, how long will the battle last, etc.

Nevertheless, the conclusions of this section do give indications of when the two models will not give the same winners. In these cases, no further evaluation of the models is needed, because it is obvious the deterministic model is not an acceptable substitute. As Springall [Ref. 28] said;

> Probably the single most important comparison is between the two predictions of which side is going to be victorious. If the two methods cannot agree on this, there is little hope for agreement on the subsidiary attributes.

B. TIME HISTORY OF EXPECTED FORCE LEVELS

If a determination has been made that the models will predict the same winner, further analysis may be made to see how well the models compare. Clark [Ref. 8] suggests

that a useful comparison is the time history of the expected number of survivors. In many models, one of the most utilized variables (performance and/or proxy variables) in measures of effectiveness for systems being evaluated by the models is the number of survivors. It has been stated previously that a bias exists in the square-law and the linear-law. Lanchester himself suspected as much; he recognized that his differential equations were approximations to the casualty rates likely to be experienced in an actual battle. Quoting from Lanchester [Ref. 19]:

> Since the forces actually consist of a finite number of finite units (instead of an infinite number of infinitesimal units) the end of the curve must show discontinuity, and break off abruptly when the last man is reached; the law based on averages evidently does not hold rigidly when the numbers become small.

Lanchester stated that his differential equations are based on averages implying an underlying stochastic process. He also suggests that his differential equations may be good approximations only so long as the force sizes are large. Although his reasoning is heuristic, Snow [Ref. 27] verified that the square-law is biased for small numbers. He showed that

$$\frac{d\bar{m}}{dt} = -a\bar{n} + a \sum_{n=1}^{n_o} nP(t,0,n) \tag{37}$$

$$\frac{d\bar{n}}{dt} = -b\bar{m} + b \sum_{m=1}^{m_o} mP(t,m,o) \tag{45}$$

The conclusion is that the solutions to Lanchester's equations for the square-law are good approximations to expected survivors so long as

$$a \sum_{n=1}^{n_o} nP(t,0,n) \approx 0 \tag{46}$$

$$b \sum_{m=1}^{m_o} mP(t,m,0) \approx 0 \tag{47}$$

Intuitively this seems sound. (46) and (47) express the boundary effects at time t. At early times in the battle, the expected force levels are near the initial force levels, and there is very little of the probability mass at the boundary. Thus the boundary effects are small. As the battle nears termination, though, much of the probability mass is near the boundary and (46) and (47) will be significant enough to effect the process. As there are no boundary effects in the deterministic process, there will be a difference in force level trajectories of the two processes.

Clark provided a derivation of (37) and (45). Additionally, he provided the following argument showing that the linear-law, which at first appears to be unbiased, is not. Through a simple derivation, it can be shown for the linear-law:

$$\frac{d\overline{m}}{dt} = - aE(MN:t) \tag{48}$$

$$\frac{d\bar{n}}{dt} = - bE(MN:t) \tag{49}$$

There are no bias terms in this expression (as opposed to (37) and (45)) since the random variable MN has a value of zero whenever M = 0 or N = 0. Consequently, the assumed differential equations for the linear-law appear unbiased.

However, even though the analysis of the solutions to these equations show that the state solution is unbiased, the time solution interjects a source of bias. This is easily seen as a result of elementary probability theory:

$$E(MN) = E(M) \cdot E(N) \quad \text{if an only if M and N are independent.}$$

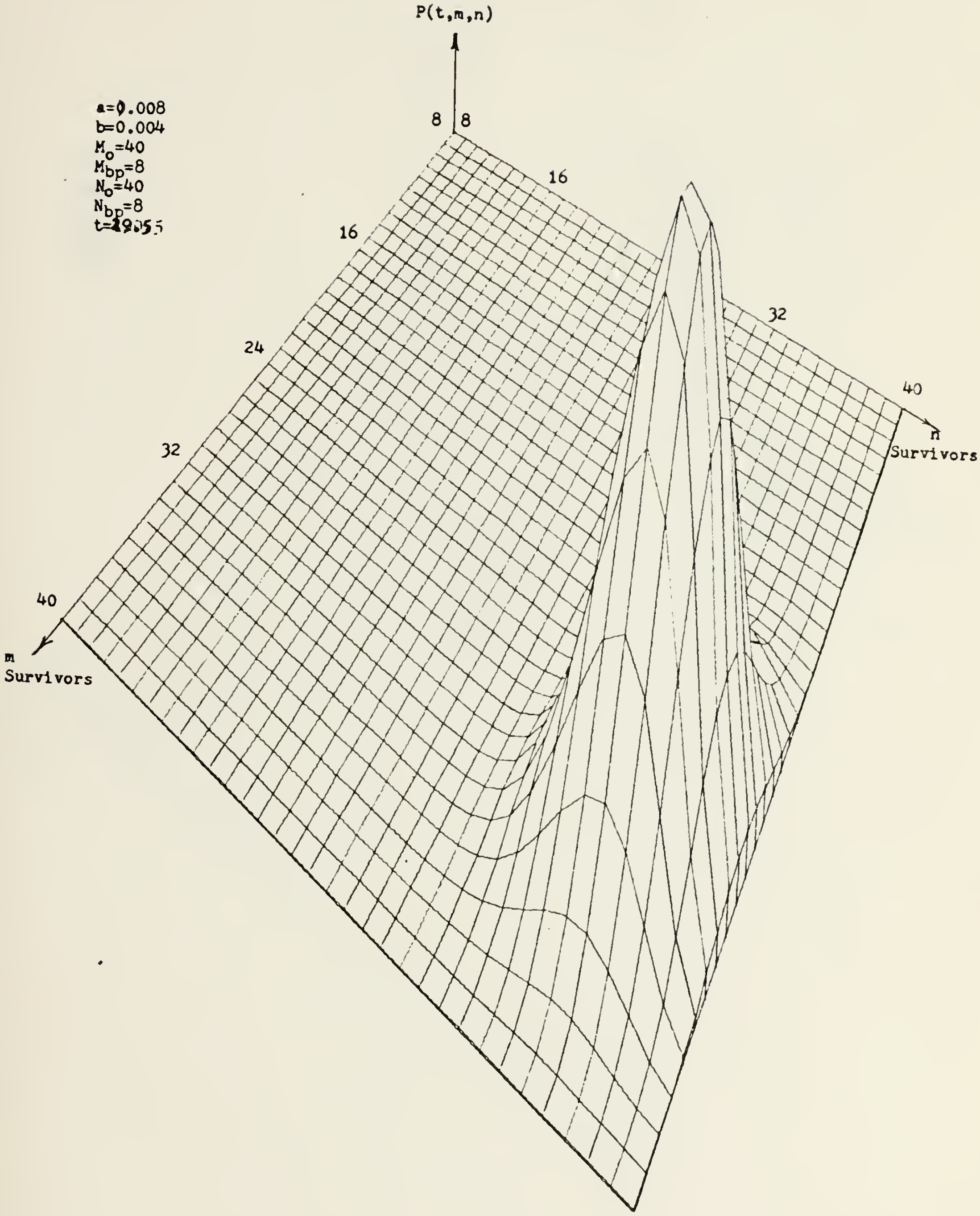
In the linear-law, they are obviously dependent on each other. The rate of change of each force level is dependent on the product of the two force levels, causing the force levels to be dependent.

Clark [Ref. 8] also gave numerical examples to show the amount of bias resulting from the difference in models. His results showed that there are significant amounts of bias resulting in both the linear-law and square-law cases. There are two things that should be noted about his results: First, his results are for small numbers of combatants; and second, his results are only for battles to annihilation. It was not shown in his dissertation that there is a

significant bias with large numbers of combatants or with breakpoints other than annihilation.

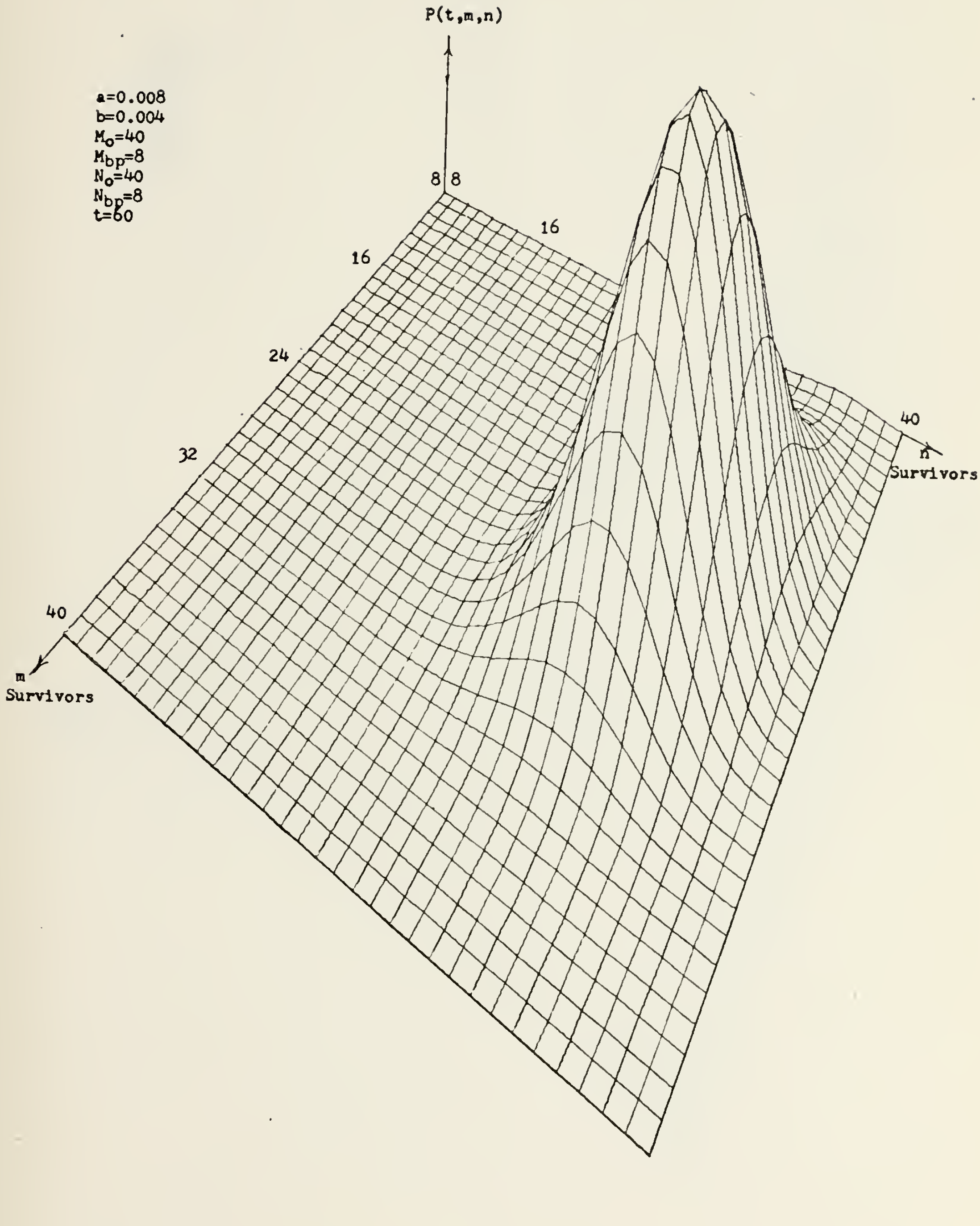
The following hypotheses are based on extension of his conclusions to include larger forces and fixed force level breakpoints. Appendix D is a derivation of the square-law stochastic attrition process showing that the bias noted by Snow exists with any set of fixed breakpoint force levels. Unfortunately, the form of the bias is not as clear as that for annihilation, since there are two additional terms. The use of the graphic aids noted earlier gives some indication that non-zero breakpoint force levels do not change the general progress of the battle. A look at the following figures will show that the time state probabilities progress in the same manner as in the case of battles to annihilation. Figures 12-15 are plots of the time state probabilities of the same model shown earlier except the breakpoint force levels are eight instead of zero (annihilation). Figures 16-19 are the same except the breakpoint force levels are sixteen. The differences are that the "wave" is absorbed by the boundary at (m_{bp},n) and (m,n_{bp}) instead of at $(0,n)$ and $(m,0)$. The question now is, What is the effect on the bias?

A first analysis of (101) and (102) might give the indication that the bias is always positive (i.e., $\overline{m}(t)-x(t) > 0$ for all $t > 0$). However, this is not true, as can be seen in the following argument.



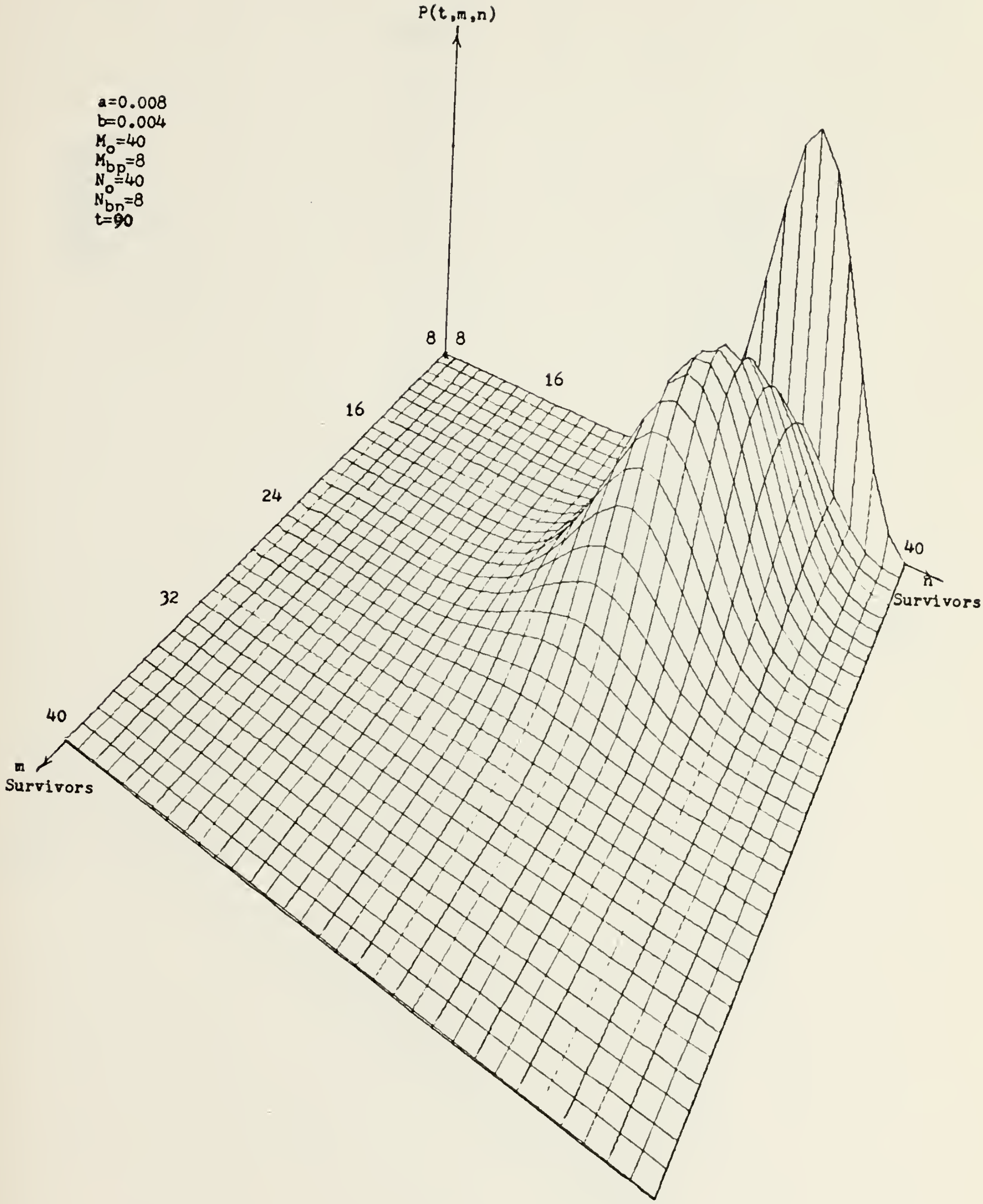
Plot of P(t,m,n) for Fixed t

FIGURE 12



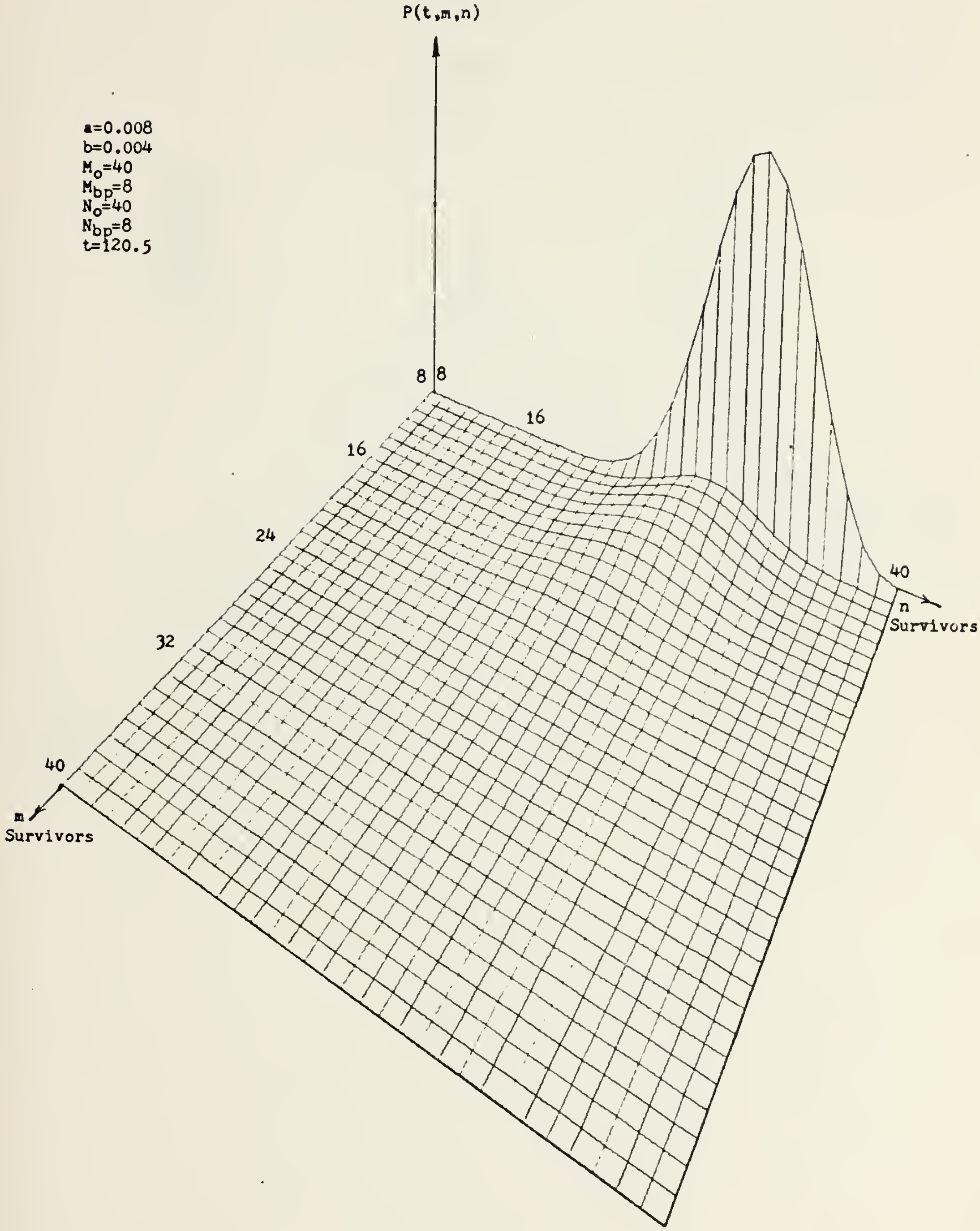
Plot of P(t,m,n) for Fixed t

FIGURE 13



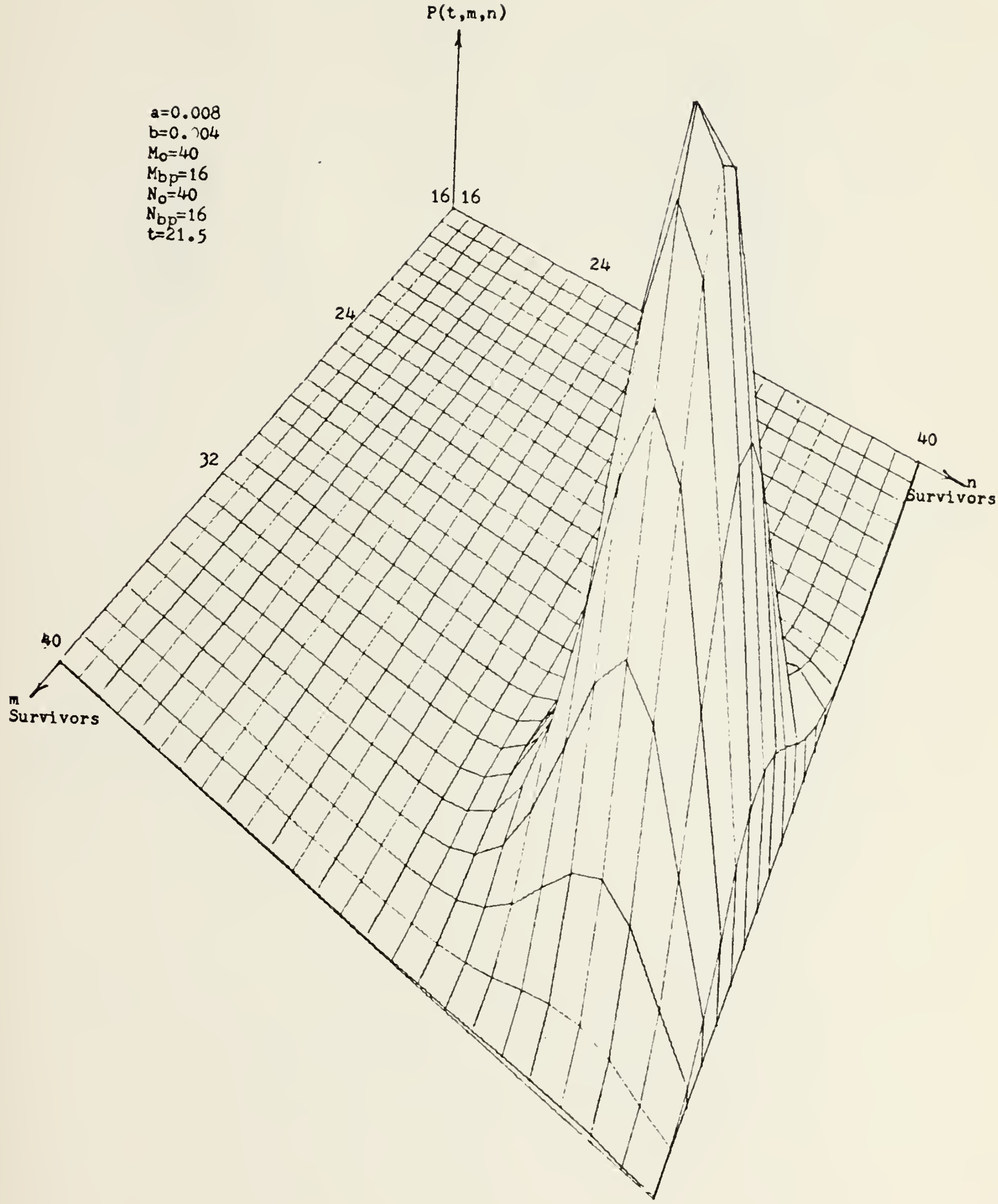
Plot of P(t,m,n) for Fixed t

FIGURE 14



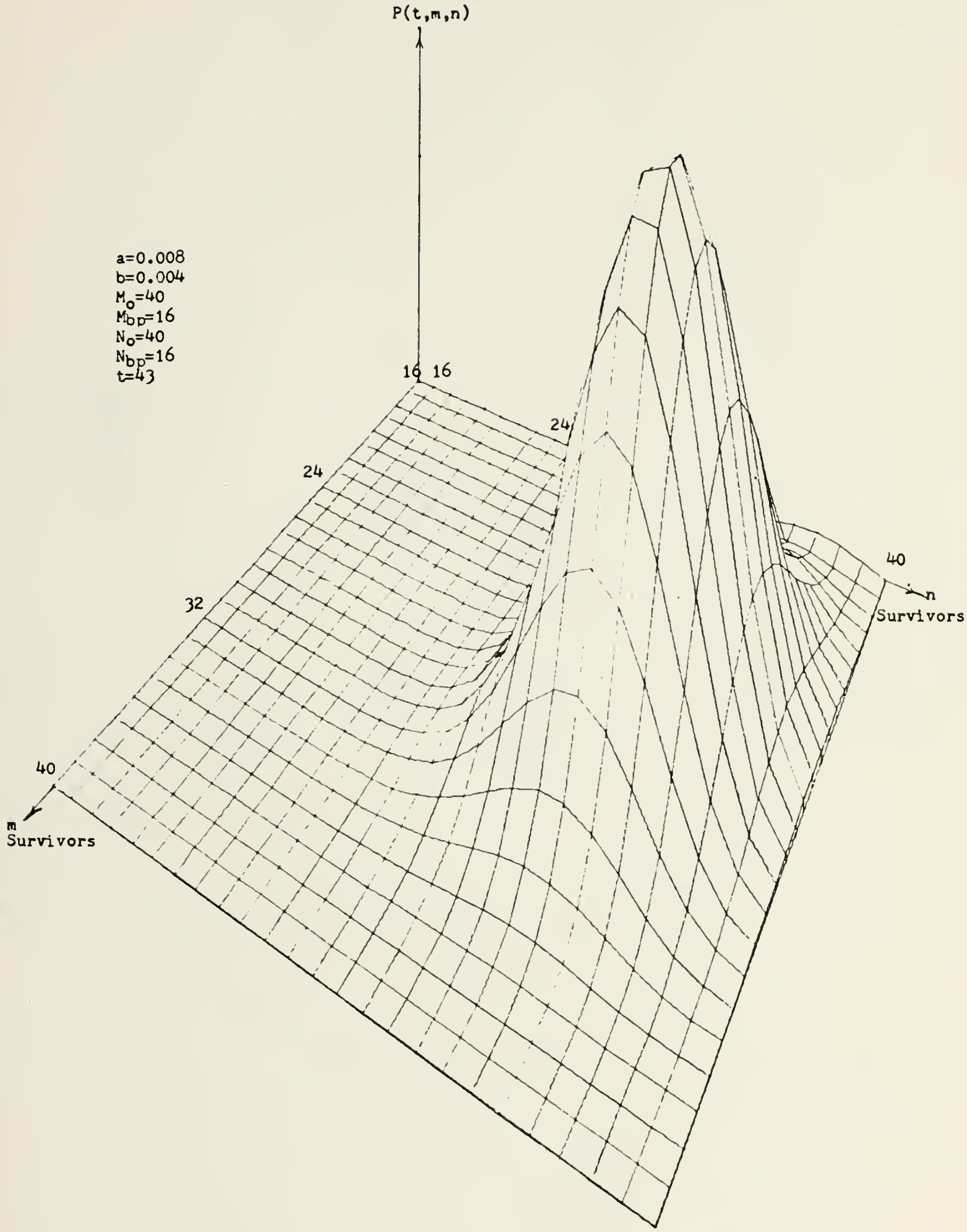
Plot of P(t,m,n) for Fixed t

FIGURE 15



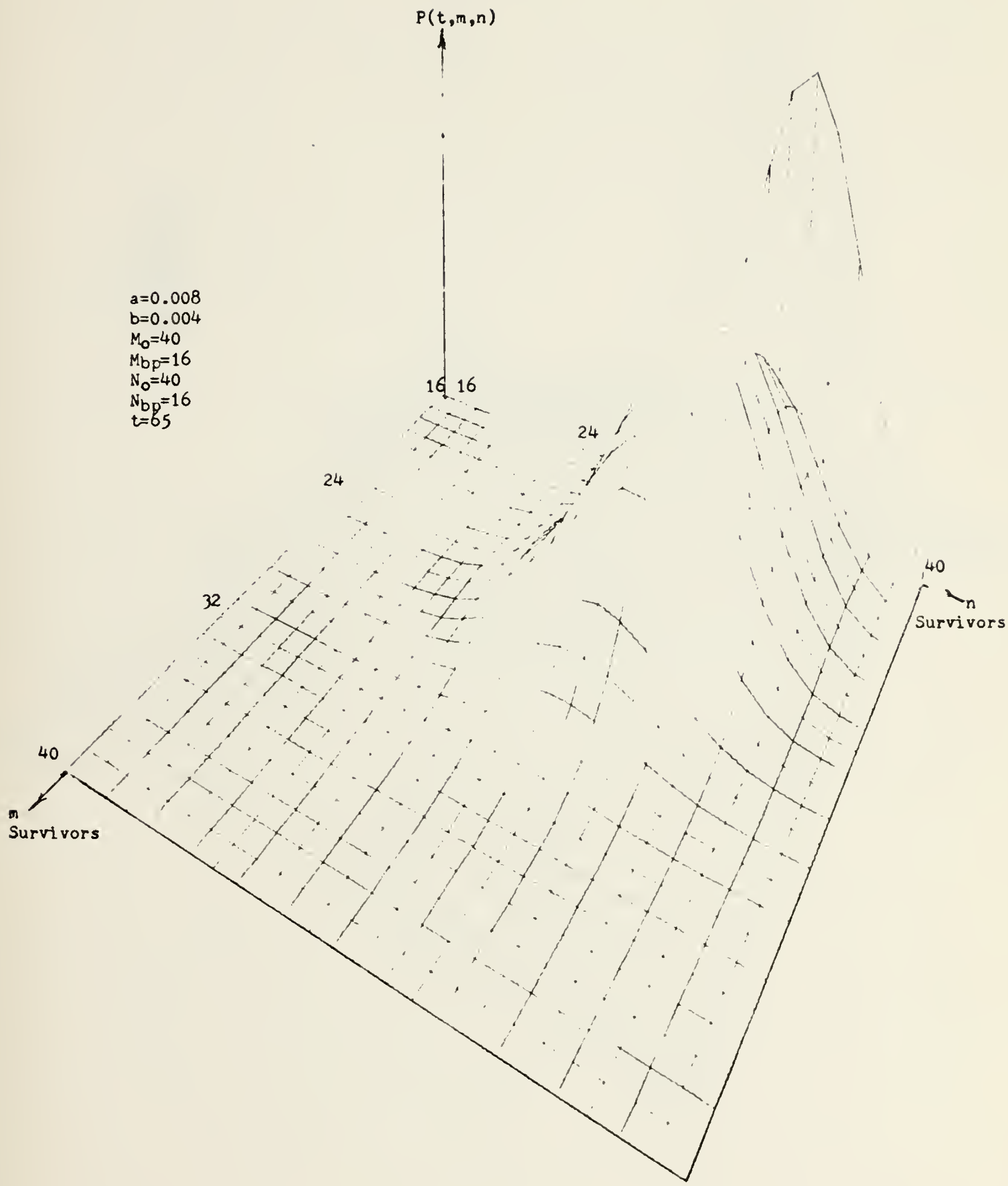
Plot of P(t,m,n) for Fixed t

FIGURE 16



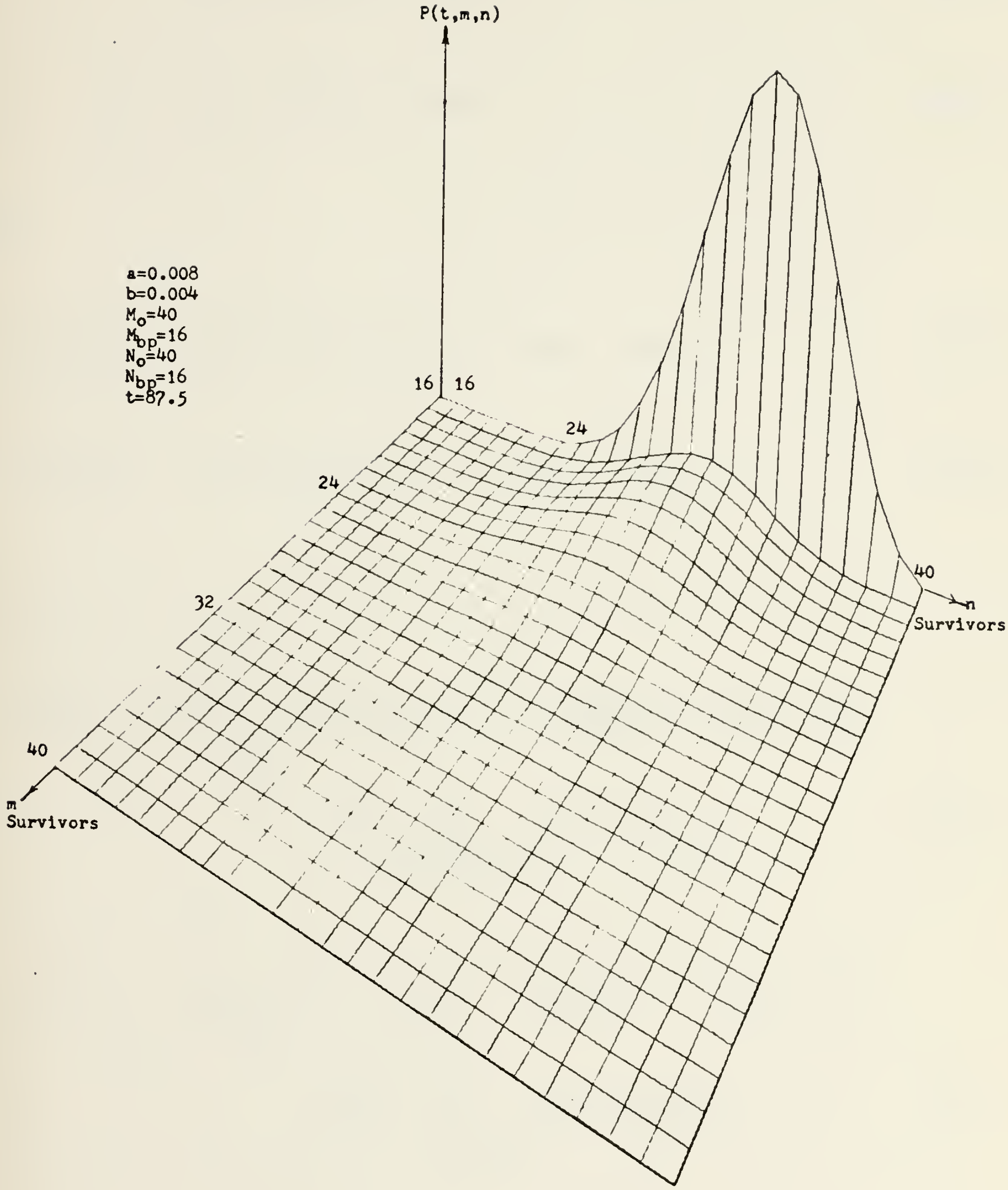
Plot of P(t,m,n) for Fixed t

FIGURE 17



Plot of P(t,m,n) for Fixed t

FIGURE 18



Plot of P(t,m,n) for Fixed t

FIGURE 19

Define

$$\Delta_X = \bar{m}(t) - x(t) \qquad (50)$$

$$\Delta_Y = \bar{n}(t) - y(t) \qquad (51)$$

then

$$\Delta_X(t=0) = \bar{m}(t=0) - x(t=0) = -an_o + an_o = 0 \qquad (52)$$

$$\Delta_Y(t=0) = \bar{n}(t=0) - y(t=0) = -bm_o + bm_o = 0 \qquad (53)$$

Let

$$S_Y(t) = a[\sum_{n=n_{bp}}^{n_o} nP(t,m_{np},n) + n_{bp}m_{bp}P(t,m_{np}+1,n_{bp}) + n_{bp}\sum_{m=m_{np}+1}^{m_o} P(t,m,n_{bp})] \qquad (54)$$

$$S_X(t) = b[\sum_{m=m_{np}}^{m_o} mP(t,m,n_{bp}) + m_{bp}n_{bp}P(t,m_{np},n_{n!}+1) + m_{np}\sum_{n=n_{bp}+1}^{n_o} P(t,m_{bp},n)] \qquad (55)$$

If

$$\frac{dx}{dt} = - ay \tag{1}$$

and

$$\frac{d\overline{m}}{dt} = - a\overline{n} + S_Y(t) \tag{101}$$

then

$$\frac{d\overline{m}}{dt} - \frac{dx}{dt} = - a\overline{n} + S_Y(t) + ay$$

$$= \frac{d(\overline{m}-x)}{dt} = - a(\overline{n}-y) + S_Y(t)$$

Substituting (50)

$$\frac{d\Delta_X}{dt} = - a\Delta_Y + S_Y(t) \tag{56}$$

and equivalently

$$\frac{d\Delta_Y}{dt} = - b\Delta_X + S_Y(t) \tag{57}$$

with initial conditions

$$\Delta_X(t = 0) = 0 \tag{52}$$

$$\Delta_Y(t = 0) = 0 \tag{53}$$

Define

$$\underset{\sim}{\Delta} = \begin{pmatrix} \Delta_X \\ \Delta_Y \end{pmatrix} \quad \text{and} \quad \underset{\sim}{U} = \begin{pmatrix} S_Y(t) \\ S_X(t) \end{pmatrix}$$

then

$$\frac{d\underset{\sim}{\Delta}}{dt} = -\begin{pmatrix} 0 & a \\ b & o \end{pmatrix}\underset{\sim}{\Delta} + \underset{\sim}{U}(t) \tag{58}$$

and the solution of (58) is given by

$$\underset{\sim}{\Delta}(t) = e^{-At}\underset{\sim}{\Delta}(t=0) + \int_0^t e^{-A(t-\tau)}\underset{\sim}{U}(\tau)\, d\tau \tag{59}$$

where

$$A = \begin{pmatrix} 0 & a \\ b & o \end{pmatrix}$$

Proof:

$$\frac{d\underset{\sim}{\Delta}}{dt} = -Ae^{-At}\underset{\sim}{\Delta}(t=0) + \underset{\sim}{U}(t) - A\int_0^t e^{-A(t-\tau)}U(\tau)d\tau - A\underset{\sim}{\Delta}(t)$$

$$= -A\underset{\sim}{\Delta}(t) + \underset{\sim}{U}(t)$$

from Sylvester's Theorem[1]

$$e^{-AT} = \begin{pmatrix} \cosh(\sqrt{ab}\ t) & -\sqrt{a/b}\ \sinh(\sqrt{ab}\ t) \\ -\sqrt{b/a}\ \sinh(\sqrt{ab}\ t) & \cosh(\sqrt{ab}\ t) \end{pmatrix}$$

Hence

$$\Delta_X(t) = \int_0^t S_Y(\tau)\cosh(\sqrt{ab}\ (t-\tau)) - S_X(\tau)\sqrt{a/b}\ \sinh(\sqrt{ab}\ (t-\tau))\,d\tau \tag{60}$$

$$\Delta_Y(t) = \int_0^t S_X(\tau)\cosh(\sqrt{ab}\ (t-\tau)) - S_Y(\tau)\sqrt{b/a}\ \sinh(\sqrt{ab}\ (t-\tau))\,d\tau \tag{61}$$

If one now takes the case where one force is annihilated ($m_{np} = n_{bp} = 0$), then $S_X(t)$ and $S_Y(t)$ simplify to

$$S_Y(t) = \sum_{n=0}^{n_o} nP(t,0,n) \tag{62}$$

$$S_X(t) = \sum_{m=0}^{m_o} mP(t,m,0) \tag{63}$$

which are simply the average number of Y survivors when X is annihilated at time t and the average number of X survivors when Y is annihilated at time t respectively. It

---

[1]Sylvester's Theorem: For a polynomial function or convergent power series of a matrix, when the matrix has distinct eigenvalues, λ_j,

$$F(A) = \sum_{i=1}^{n} F(\lambda_i) \sum_{\substack{j=1 \\ j\neq i}}^{n} \left(\frac{A-\lambda_j I}{\lambda_i - \lambda_j}\right) \text{ where A is an nxn matrix.}$$

seems intuitively appealing that if X wins decisively, $S_Y(t) \to 0$. If this is true, then (60) will become negative and (61) will become positive. This seems to be confirmed by numerical results obtained.

Another interesting case to examine is the case of complete parity ($a=b$, $m_o=n_o$, $m_{np}=n_{bp}$) where $m_{np}=n_{bp}=0$. In this case, $S_Y(t) = S_X(t) = S(t)$ and

$e^{-\sqrt{ab}(t-\tau)}$

$$\Delta_X(t) = \int_0^t S(\tau)[\cosh(\sqrt{ab}\,(t-\tau)) - \sinh(\sqrt{ab}\,(t-\tau))]d\tau \tag{64}$$

$$\Delta_Y(t) = \int_0^t S(\tau)[\cosh(\sqrt{ab}\,(t-\tau)) - \sinh(\sqrt{ab}\,(t-\tau))]d\tau \tag{65}$$

By inspection, $\Delta_X(t) > 0$ and $\Delta_Y(t) > 0$ for $t > 0$, and both biases are positive. This is reinforced by numerical results.

The general conclusion reached from this analysis is that the biases are both positive unless one side is going to win very decisively. In this case, the bias on the winner's side will be negative, while the loser's is positive.

The remainder of this section will be a statement and discussion of hypotheses concerning bias.

Hypothesis 2-1

> Given fixed initial force levels and fixed attrition coefficients, as the breakpoint force levels increase, the numerical bias decreases. But as a percentage of casualties of the deterministic model, the bias increases.

This hypothesis has heuristic appeal when the bias equations are analyzed. The bias terms are (see Appendix D):

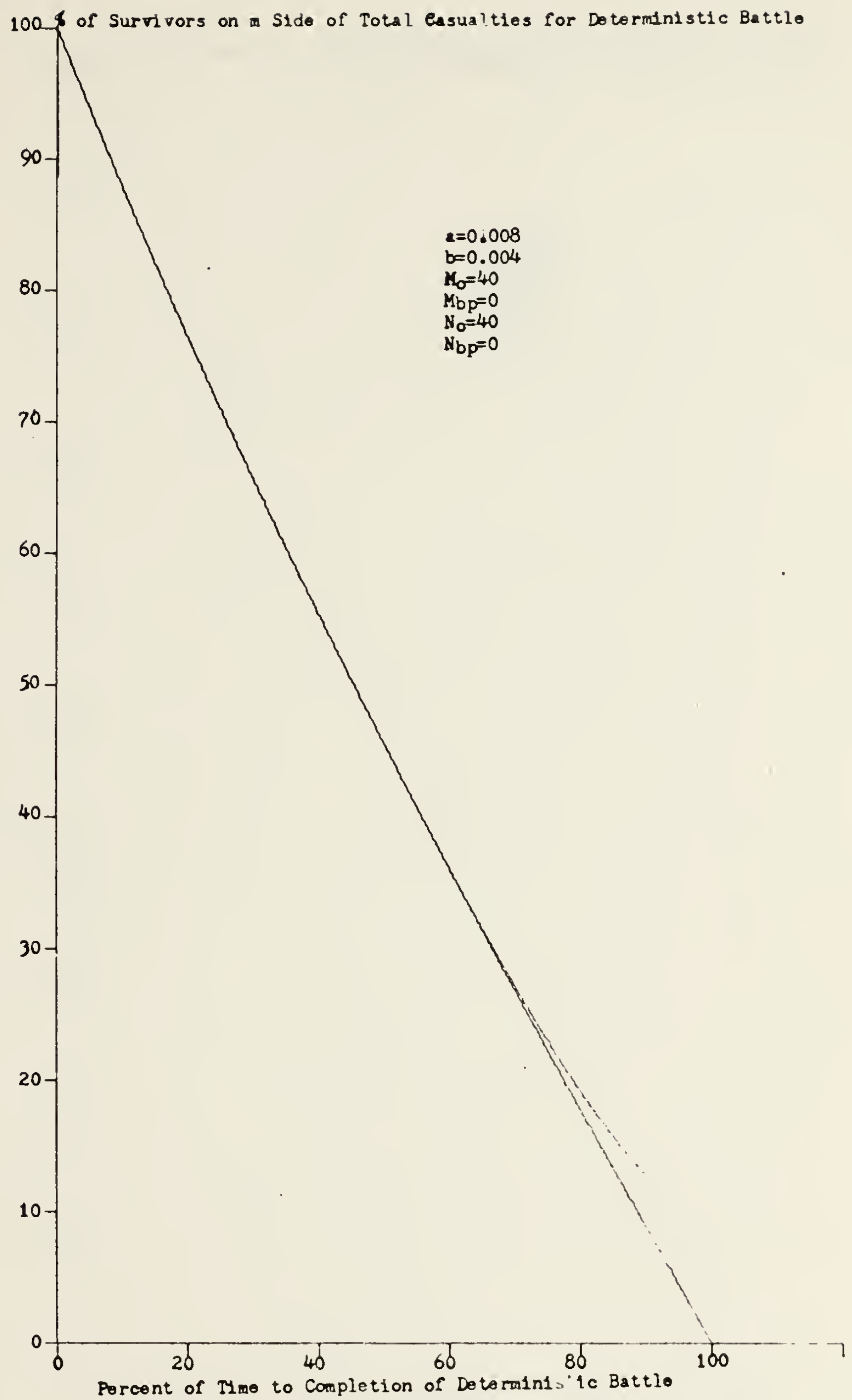
$$S_Y(t) = a\left[\sum_{n=n_{bp}}^{n_o} nP(t,m_{np},n) + n_{bp}m_{bp}P(t,m_{np}+1,n_{bp}) + n_{bp}\sum_{m=m_{np}+1}^{m_o} P(t,m,n_{bp})\right] \tag{54}$$

$$S_X(t) = b\left[\sum_{m=m_{bp}}^{m_o} mP(t,m,n_{bp}) + m_{np}n_{bp}P(t,m_{bp},n_{bp}+1) + m_{bp}\sum_{n=n_{bp}+1}^{n_o} P(t,m_{np},n)\right] . \tag{55}$$

As n_{bp} and m_{bp} increase, the number of terms in each summation decreases. Although the second and third terms are multiplied by n_{bp} (which is increasing), it would seem that this may be more than offset by the decrease in number of terms.

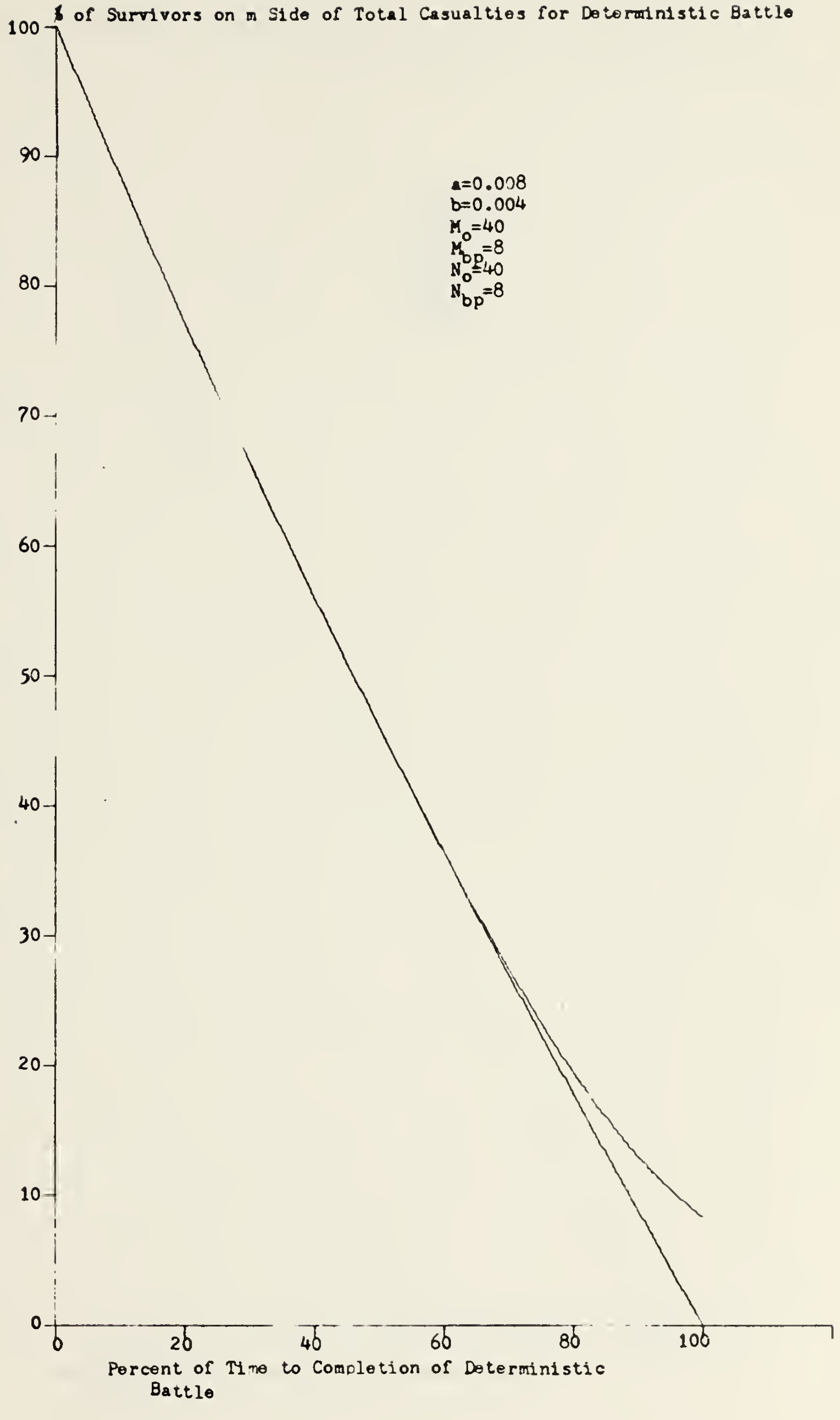
Even if this is true, as a percentage, the bias will increase unless the actual bias decreases as fast as $m_o - m_{bp}$. Although parametric analysis cannot logically prove the truth of this statement, it shows that the statement is certainly plausible.

As all parameters except m_{bp} and n_{bp} are fixed, a numerical parametric analysis can be performed. A range of fixed parameters was chosen; small and medium values of m_o and n_o, and several combinations of a and b. In all cases the results generally agreed. The following figures illustrate the results. They plot $\Delta_X\%$ and $\Delta_Y\%$ as defined in



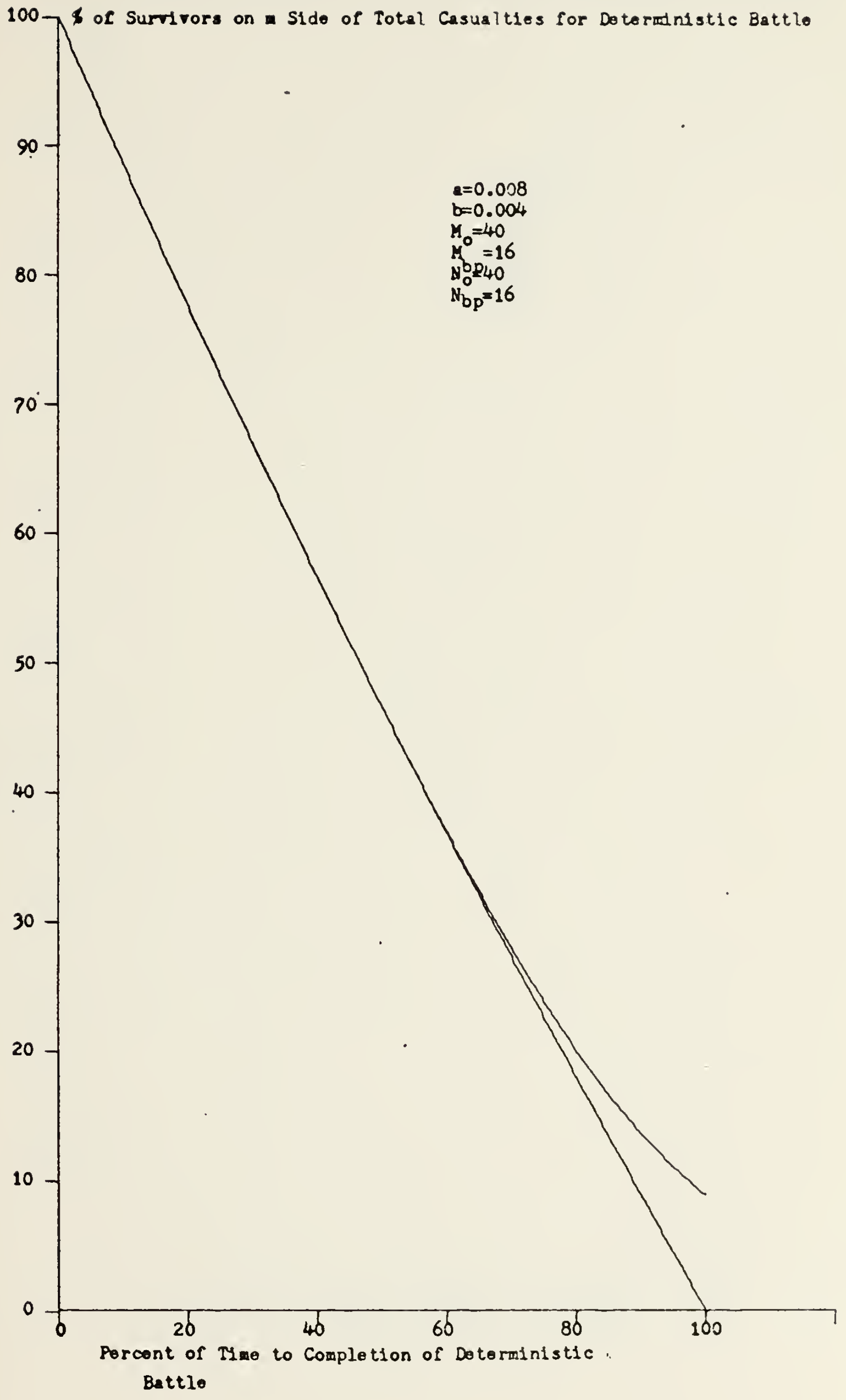
Time History Showing Bias

FIGURE 20



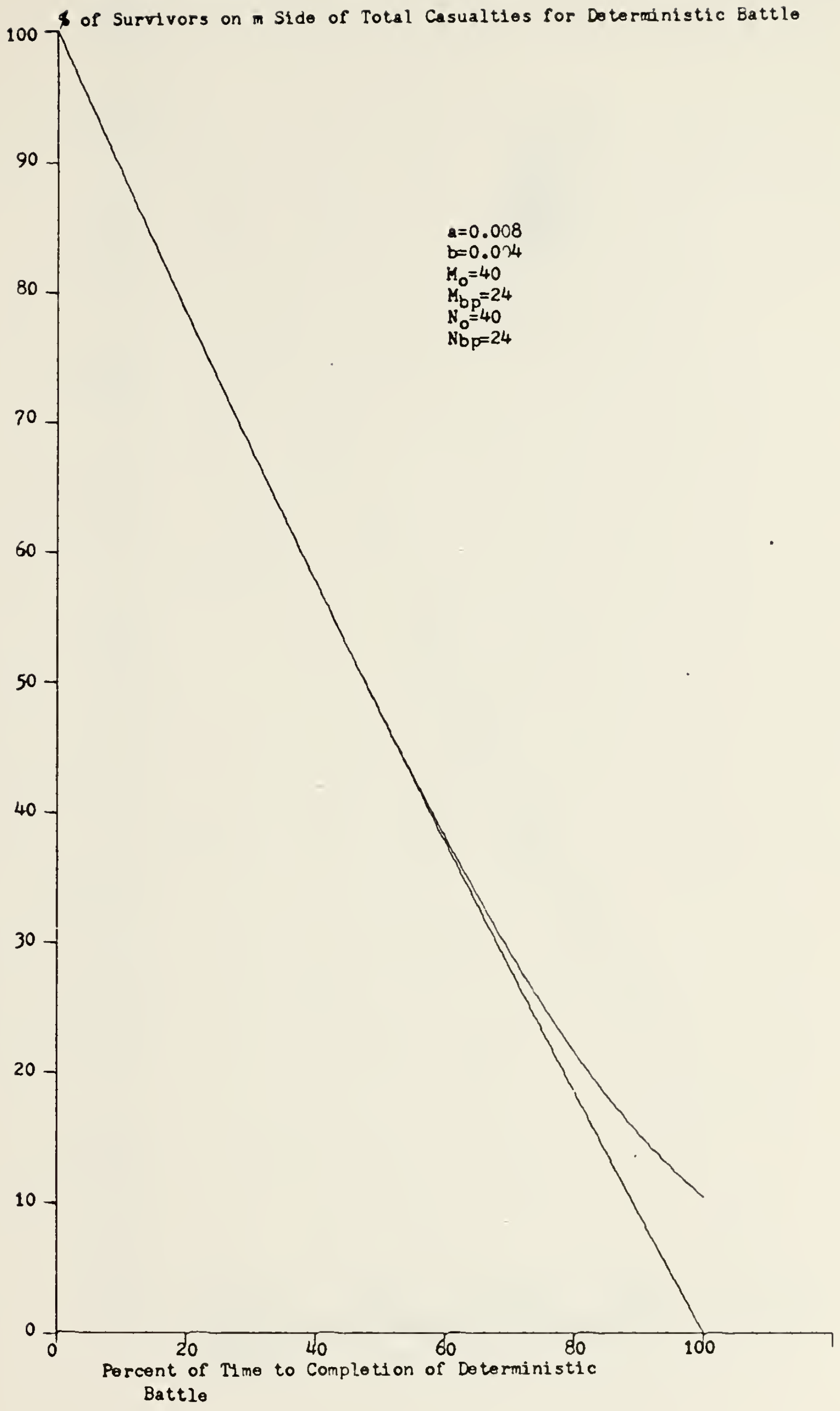
Time History Showing Bias

FIGURE 21



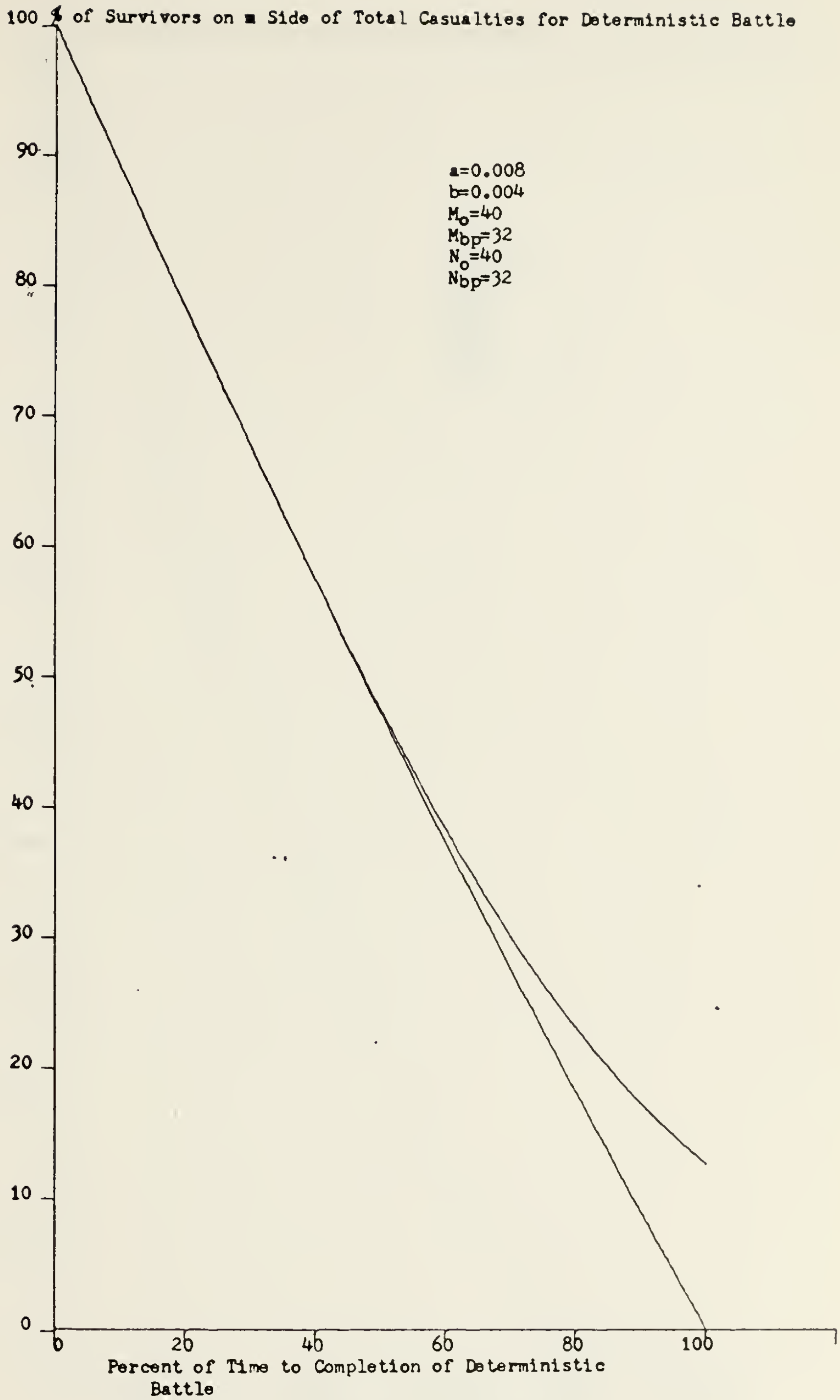
Time History Showing Bias

FIGURE 22



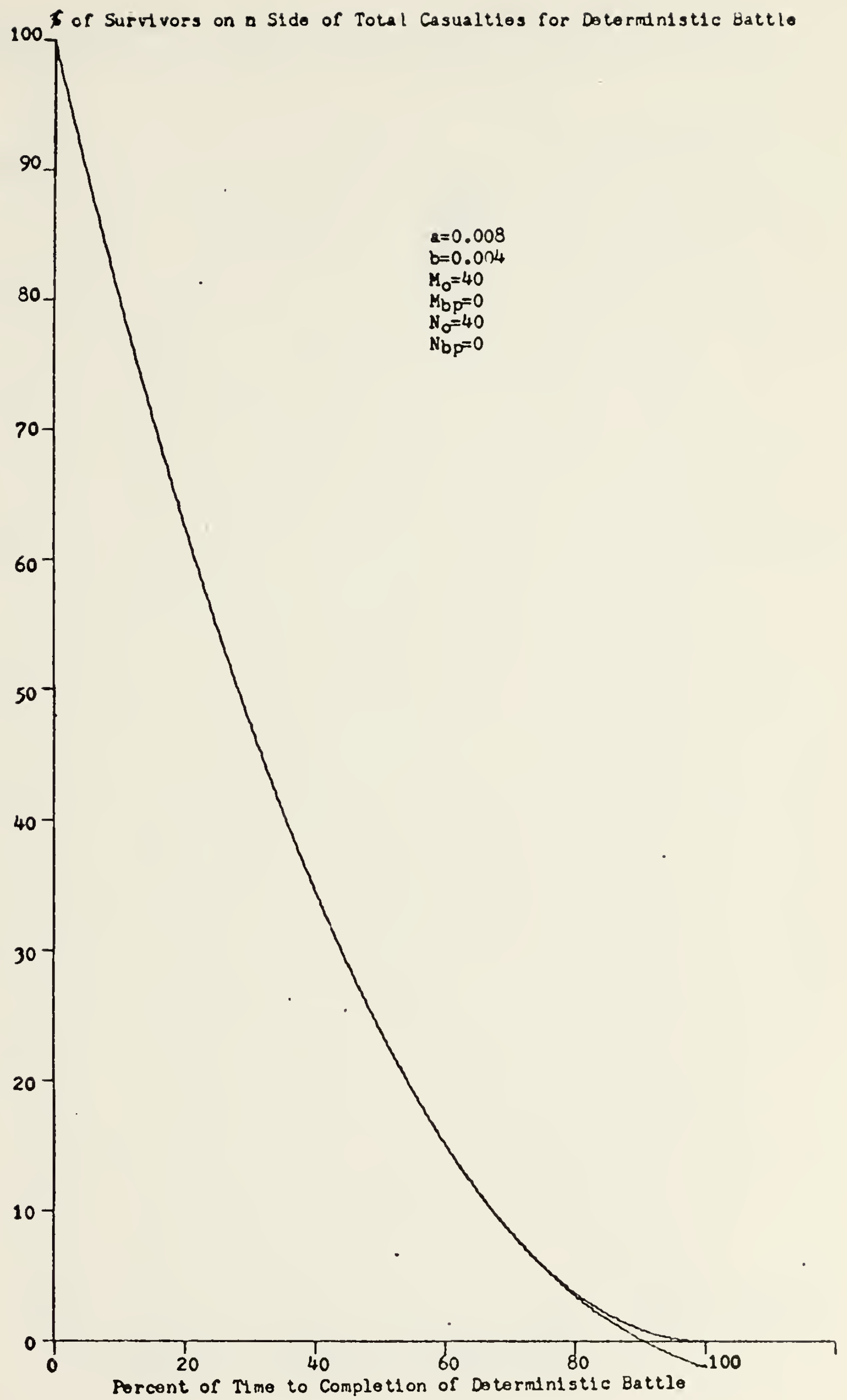
Time History Showing Bias

FIGURE 23



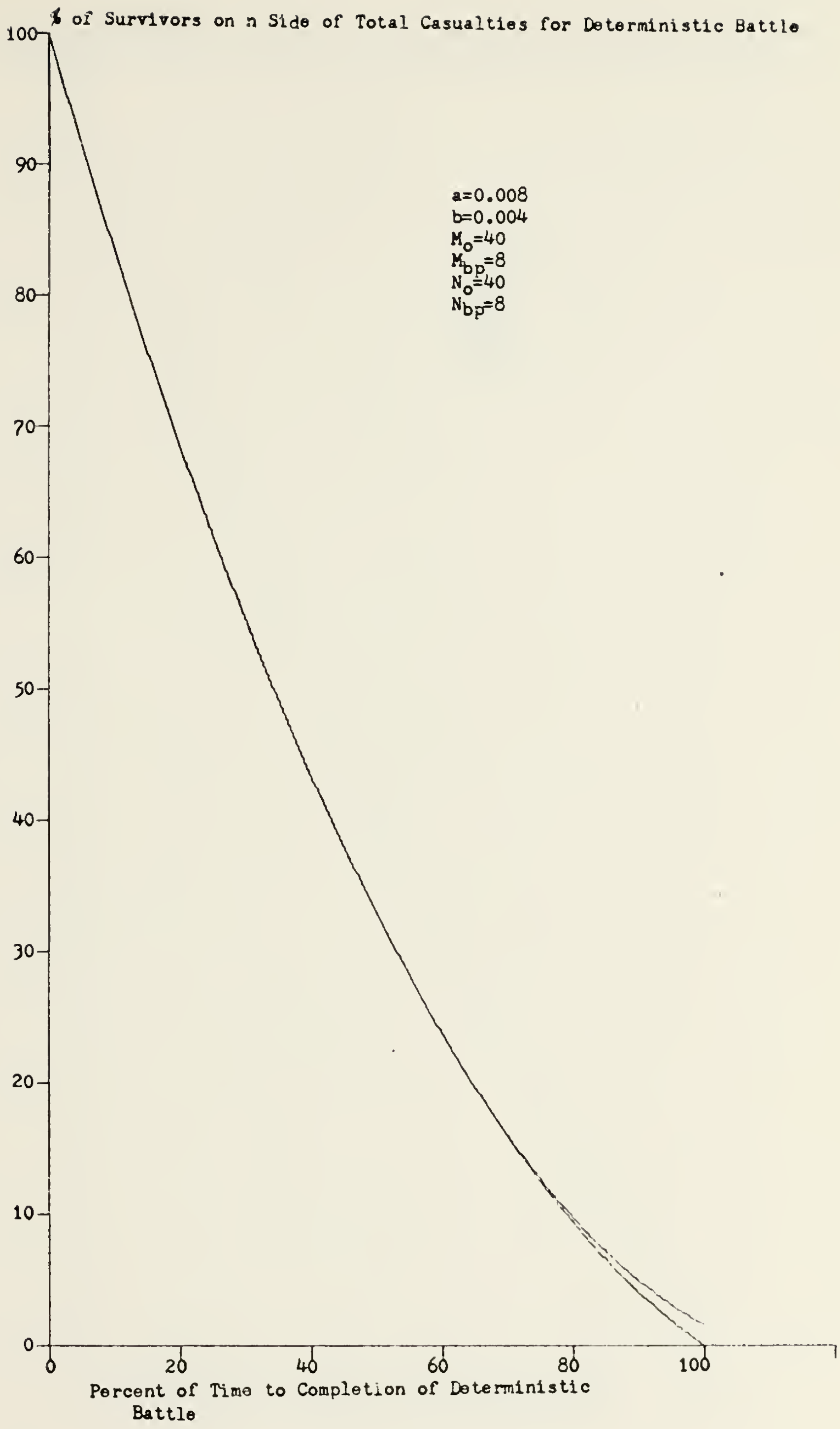
Time History Showing Bias

FIGURE 24



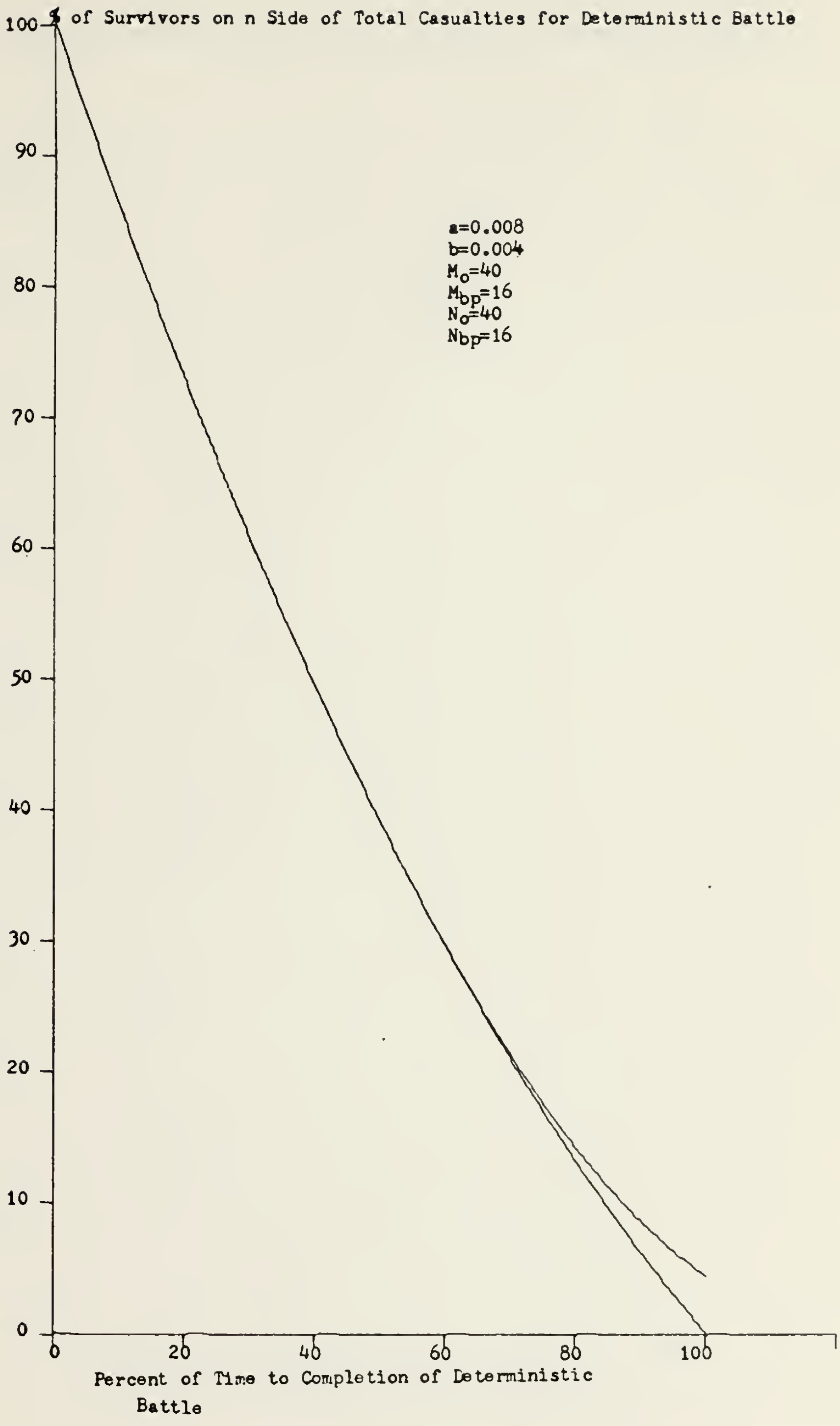
Time History Showing Bias

FIGURE 25



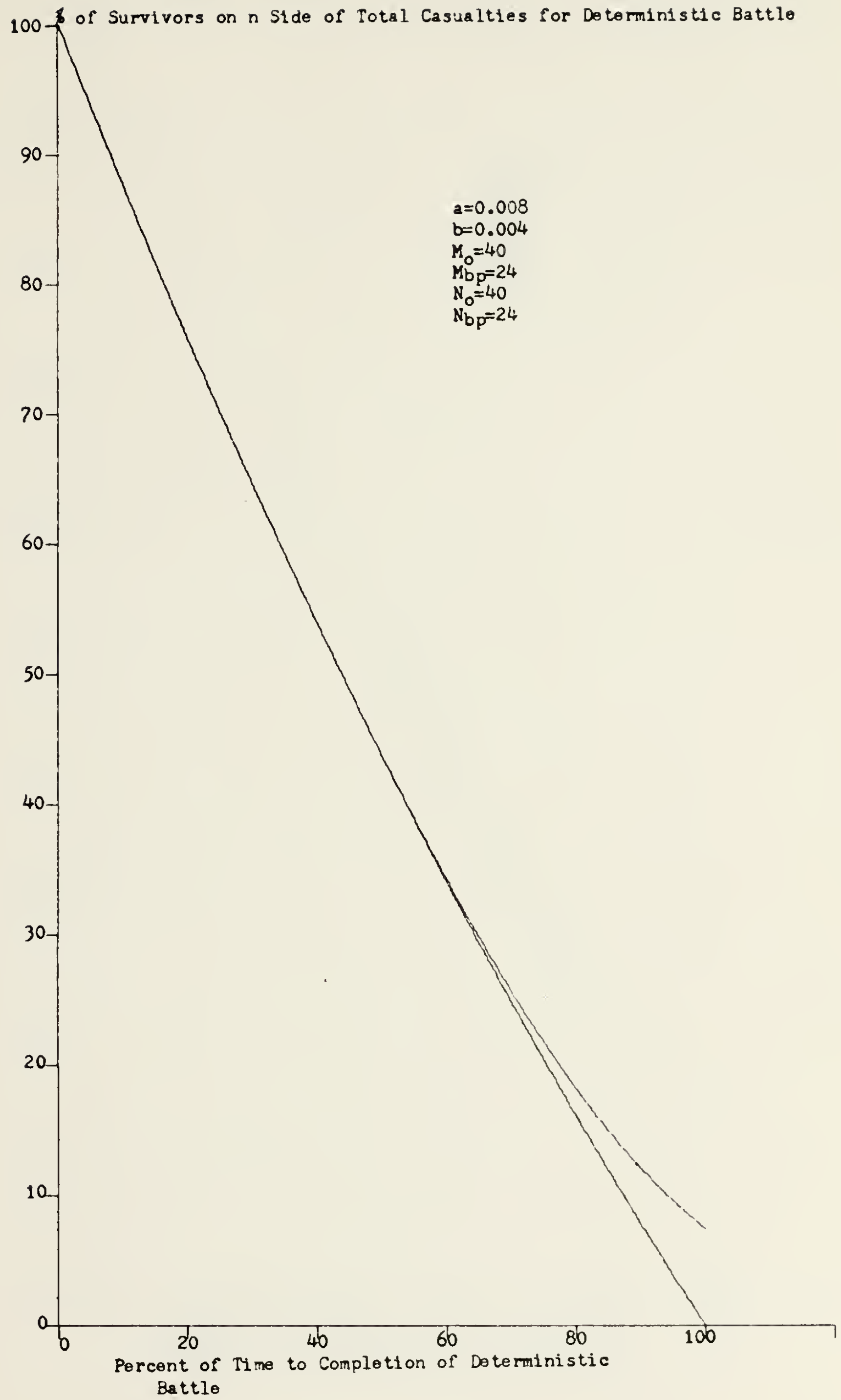
Time History Showing Bias

FIGURE 26



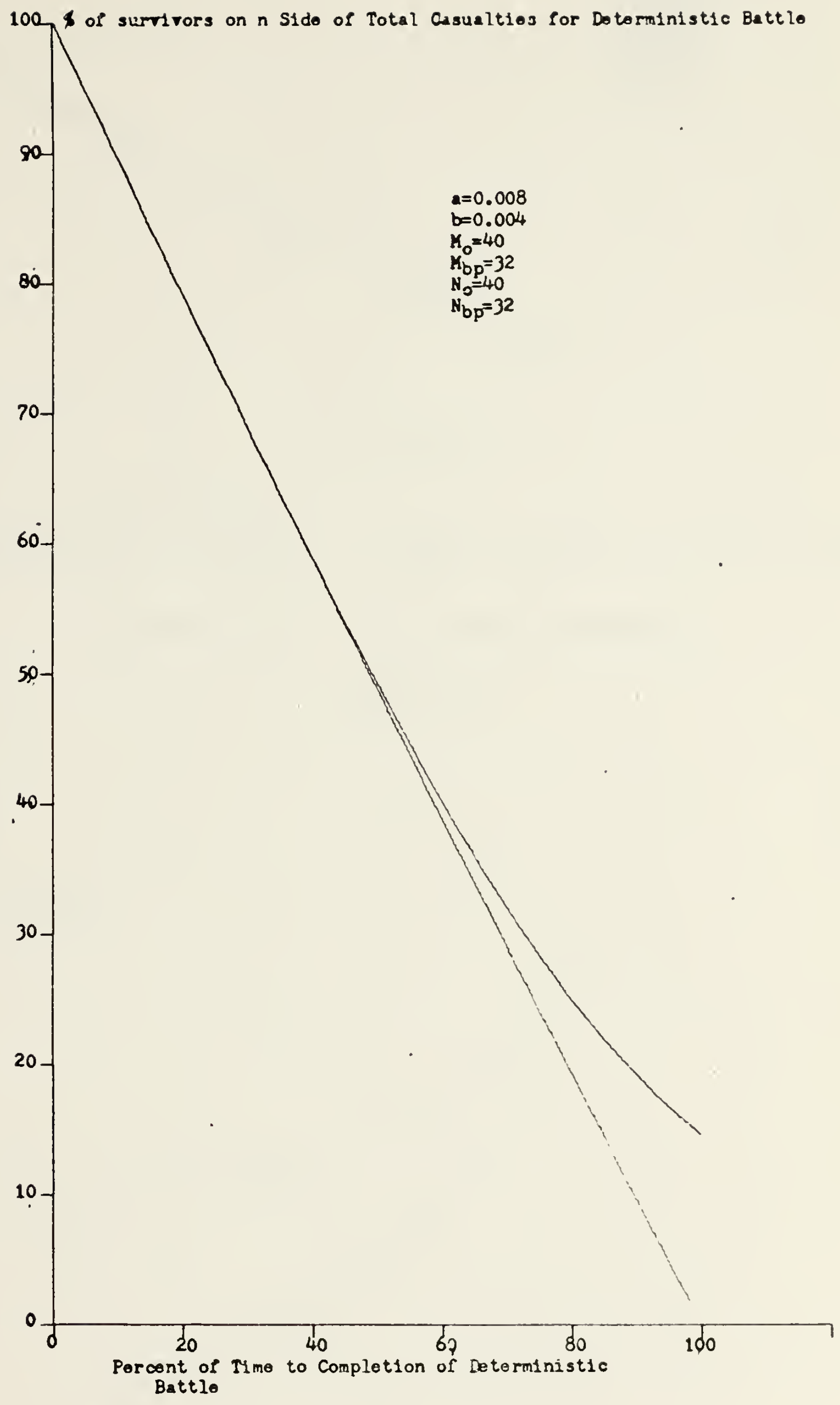
Time History Showing Bias

FIGURE 27



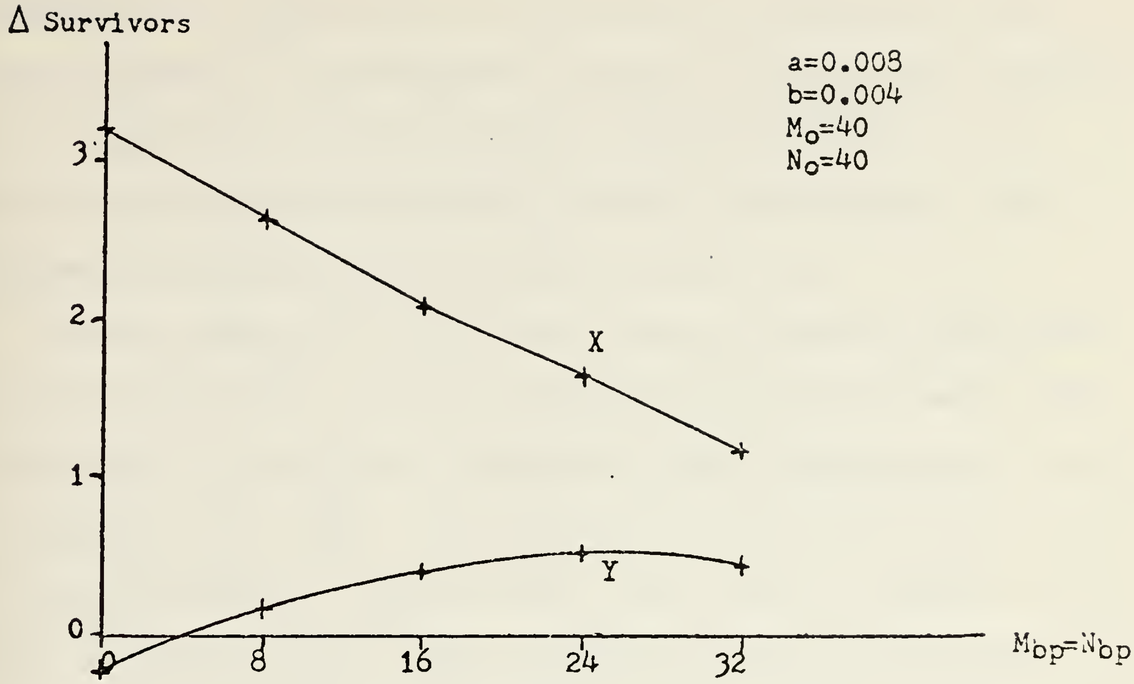
Time History Showing Bias

FIGURE 28



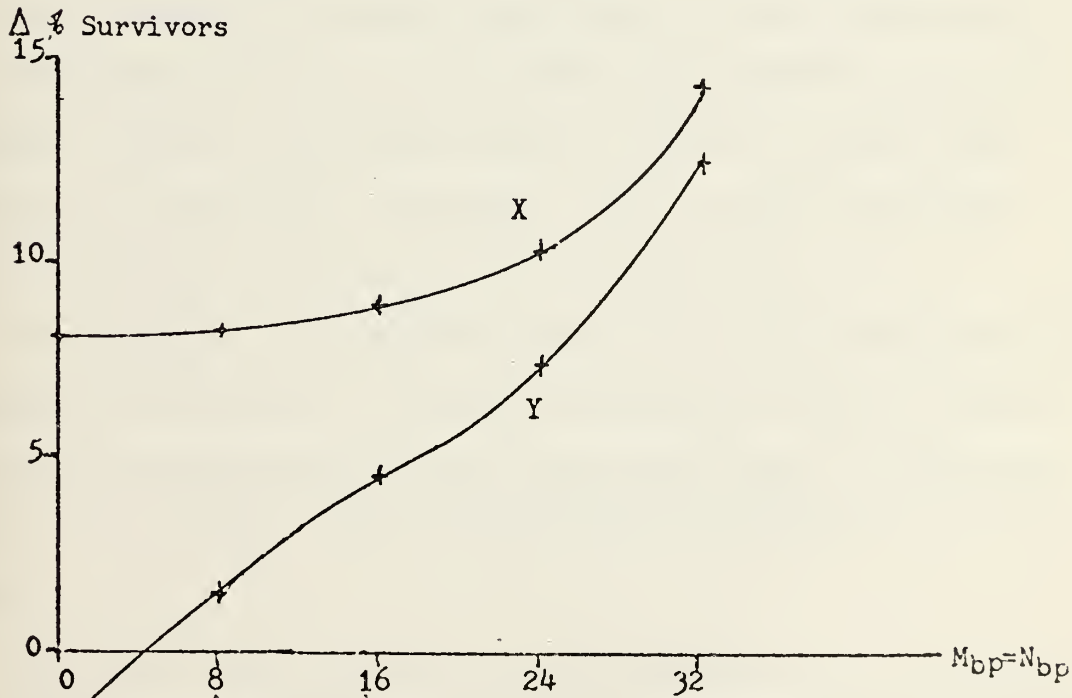
Time History Showing Bias

FIGURE 29



Δ_X and Δ_Y for Increasing Breakpoints

FIGURE 30



Δ_X and Δ_Y for Increasing Breakpoints

FIGURE 31

Appendix C as functions of scaled time as defined in the same appendix. The important things to notice are when the lines separate and how large the final separation is for the various input parameters listed on each figure. The figures are summarized in Figures 30 and 31.

In our examples, X is taken always to be the loser and Y to be the winner. As the breakpoints are increased, the difference in X survivors decreases from slightly over three to slightly over one. This seems to confirm the first part of the hypothesis. Additionally, the rate of change of the bias is less than the rate of change of the breakpoints, so the percentage of bias increases, as indicated in Figure 31. This seems to confirm the second part of the hypothesis. However, the bias for the winning side does not follow the hypothesis (Y wins very decisively in this case). Indeed, it starts negative, becomes positive, and then decreases. But two things must be noted before the hypothesis is rejected: First, the slope of the Δ_Y curve is small; and second, it eventually decreases. This leads to the $\Delta_Y\%$ curve in Figure 31, where $\Delta_Y\%$ is increasing.

It is argued in Appendix C that $\Delta\%$ is the more important measure of the bias. If this is accepted, then the second half of the hypothesis is more important and, as indicated above, it is true. All of the results in Table 2 reinforce this.

One final way of stating this conclusion is; the greater the differences m_o-m_{bp} and n_o-n_{bp}, the less significant the biases.

$\underline{a}$	$\underline{b}$	$\underline{m_o}$	$\underline{n_o}$	$\underline{m_{bp}}$	$\underline{n_{bp}}$	$\underline{\Delta_X}$	$\underline{\Delta_X\%}$	$\underline{\Delta_Y}$	$\underline{\Delta_Y\%}$
0.008	0.004	40	40	0	0	3.22	8.05	-0.23	- 1.96
0.008	0.004	40	40	8	8	2.62	8.19	0.18	1.61
0.008	0.004	40	40	16	16	2.11	8.79	0.42	4.40
0.008	0.004	40	40	24	24	1.64	10.25	0.52	7.41
0.008	0.004	40	40	32	32	1.15	14.38	0.47	12.43
0.008	0.004	24	24	0	0	2.48	10.33	-0.22	- 3.13
0.008	0.004	24	24	4	4	2.10	10.50	0.06	0.88
0.008	0.004	24	24	9	9	1.68	11.20	0.29	4.93
0.008	0.004	24	24	14	14	1.31	13.10	0.38	8.74
0.008	0.004	24	24	19	19	0.94	18.80	0.38	16.10
0.004	0.0015	40	40	16	16	2.06	8.58	0.28	4.06
0.004	0.0015	40	40	24	24	1.62	10.13	0.36	7.02
0.004	0.0015	40	40	32	32	1.12	14.00	0.33	11.79
0.004	0.0015	24	24	0	0	2.28	9.50	0.13	- 2.58
0.004	0.0015	24	24	4	4	1.98	9.90	0.04	0.82
0.004	0.0015	24	24	9	9	1.63	10.87	0.19	4.48
0.004	0.0015	24	24	14	14	1.28	12.80	0.27	8.49
0.004	0.0015	24	24	19	19	0.90	18.00	0.25	14.37

TABLE II. Bias for Different Breakpoints

Hypothesis 2-2

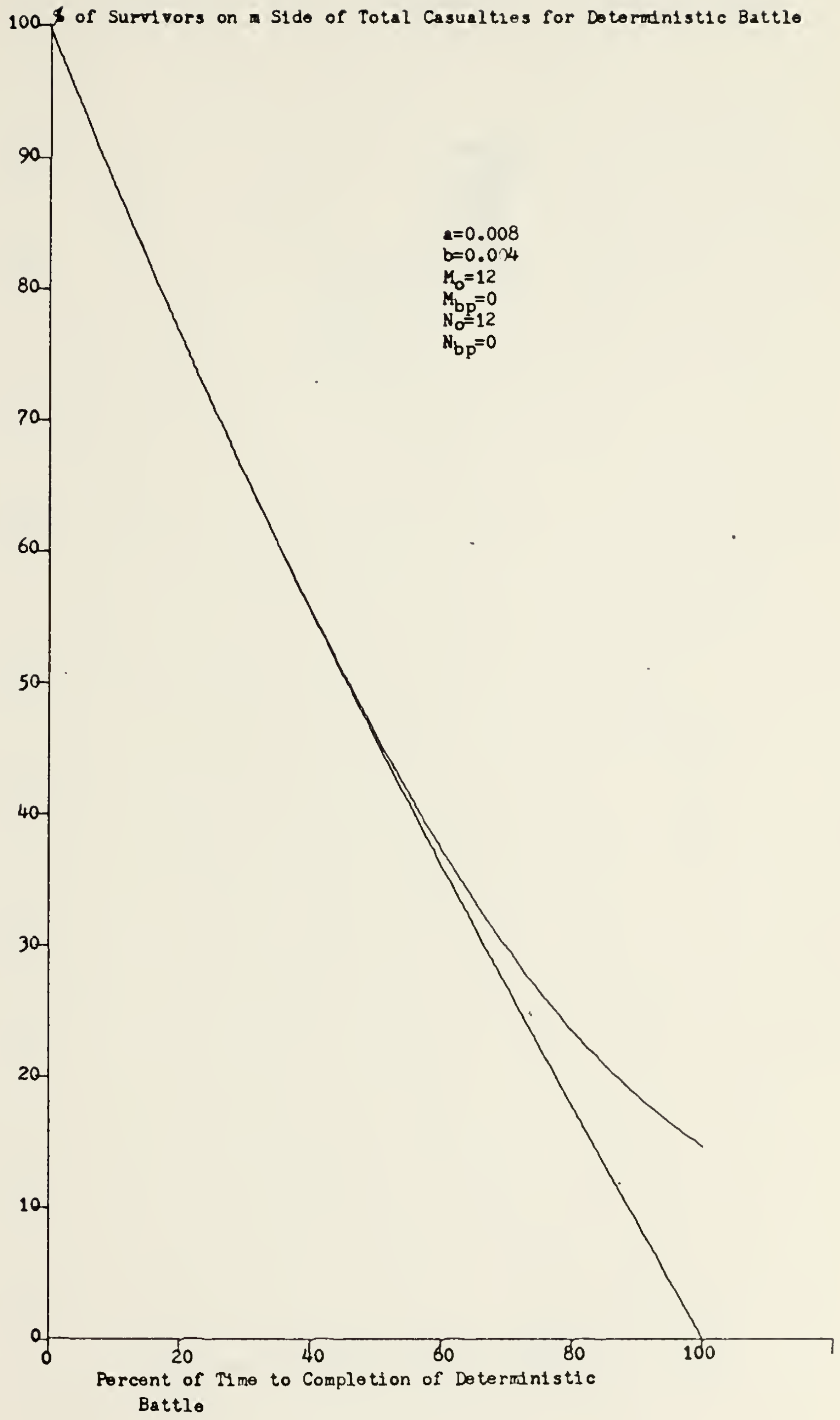
> For fixed F_X and F_Y, and fixed attrition rate coefficients, the larger the initial forces, the greater the numerical bias, but the smaller the percentage bias.

A generally accepted conclusion for Lanchester-type models is a type of "law of large numbers". For a battle to annihilation, the differences between a deterministic model and the equivalent stochastic model "disappear" as the initial force levels increase. Hypothesis 2-2 is a more general statement of this not for battles to annihilation, but to fixed casualty/initial force level breakpoints. Additionally, it is concerned only with the differences in force level histories. Hypothesis 1-2 was a discussion of the "law of large numbers" for the probability of winning.

There is an intuitive appeal for the hypothesis. The time history of the force levels for a deterministic model is something of an "average" time history of the stochastic model. As the number of combatants involved increases, the greater the aggregation, and the closer this average is to the mean force level history of the stochastic model.

All but two parameters are held constant, so a numerical parametric analysis can easily be done. Figures 31 to 35 show some of the results of the variation in parameters. Figures 37 and 38 summarize them and show that, for these fixed parameters, the hypothesis is true.

Graphs of the results in Table III would give similar results and further reinforce the hypothesis.

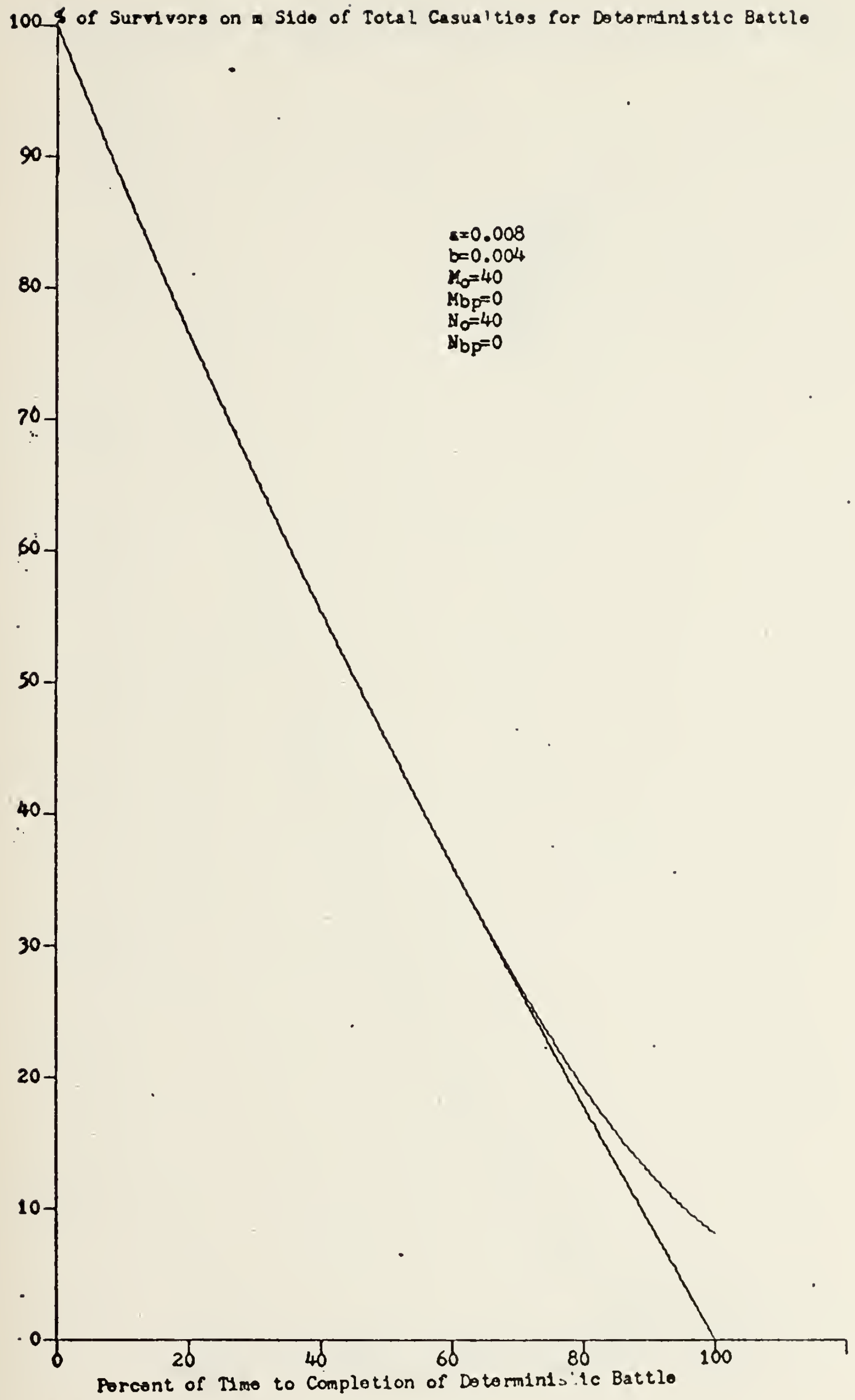


Time History Showing Bias

FIGURE 32

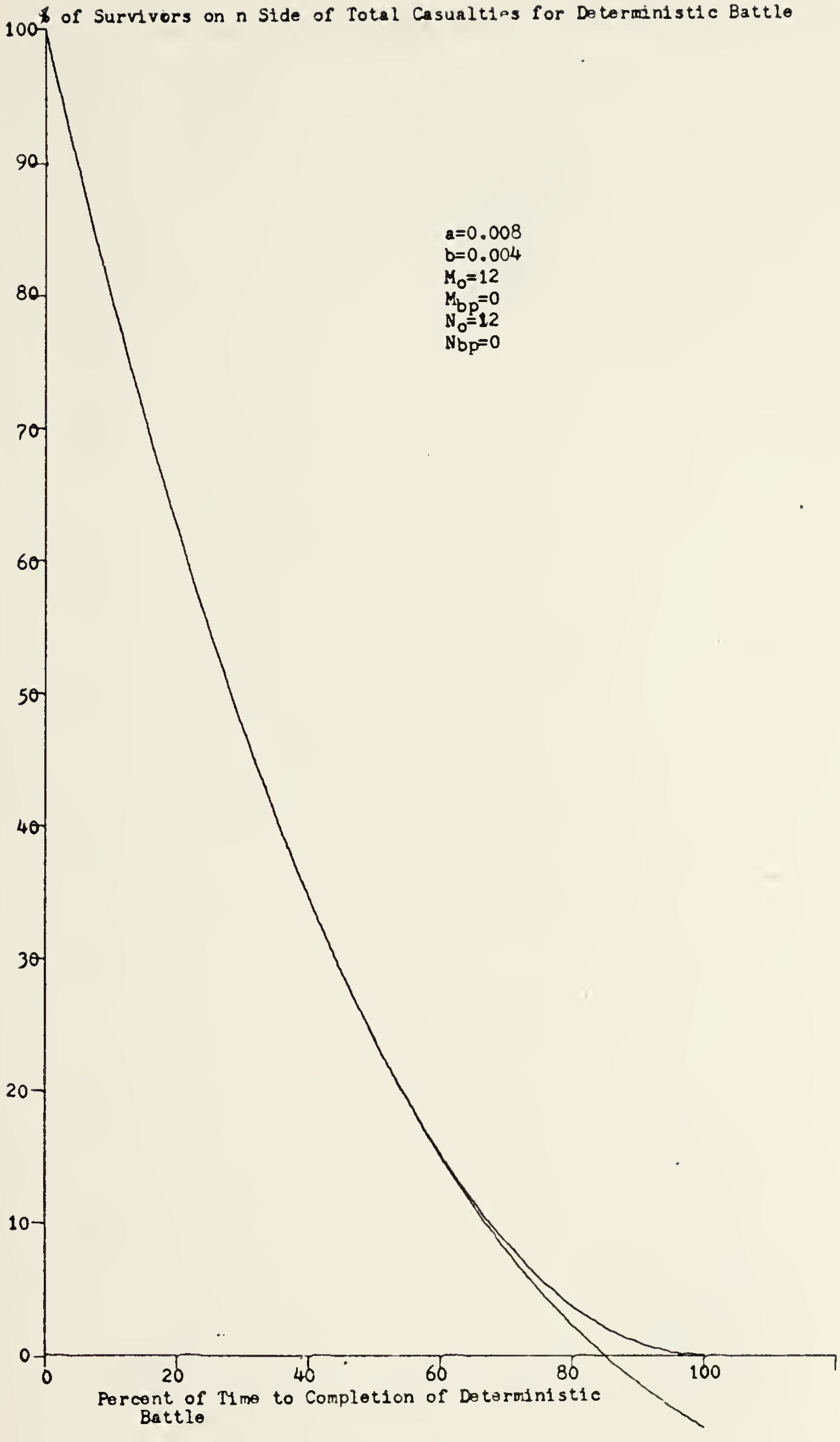
Time History Showing Bias

FIGURE 33



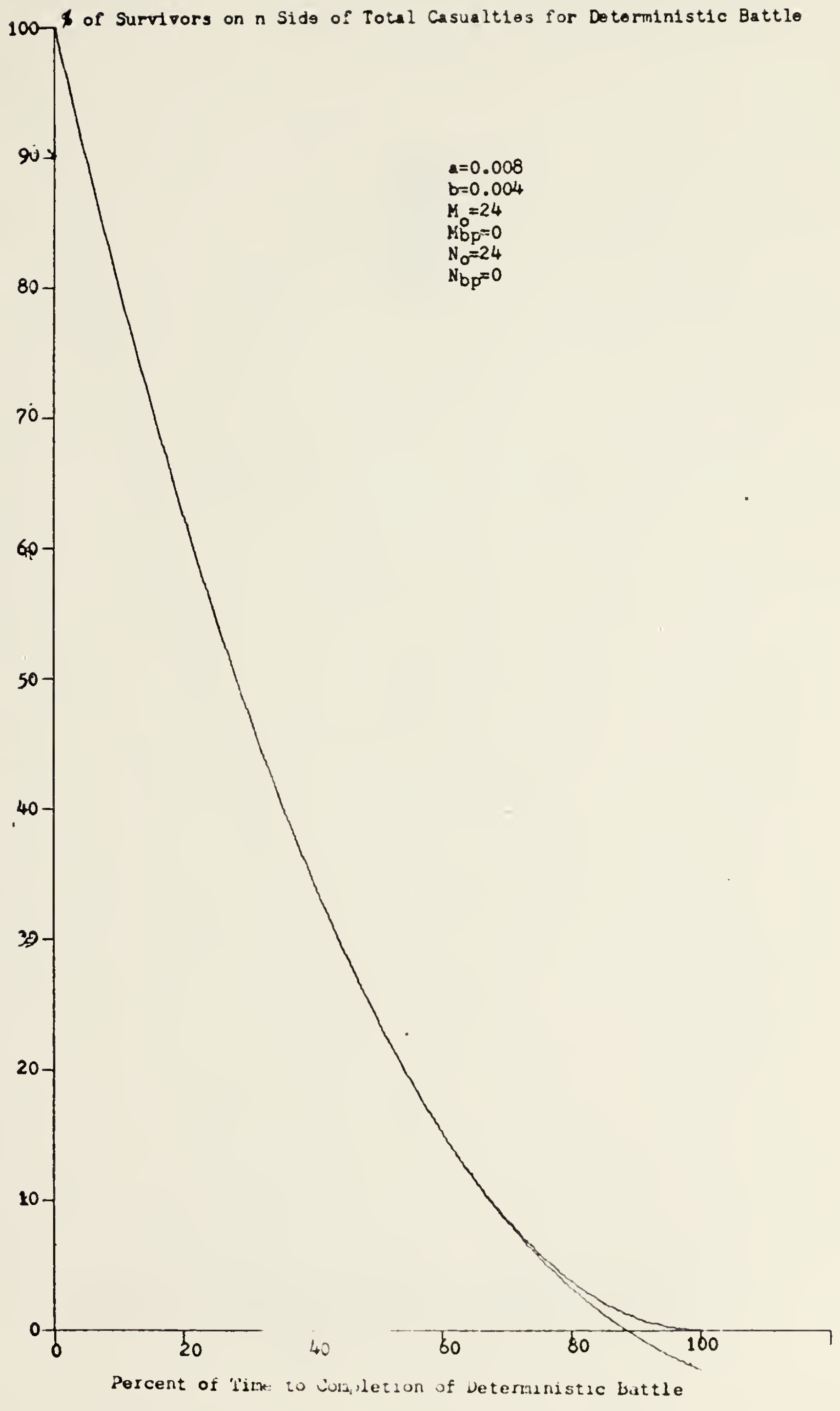
Time History Showing Bias

FIGURE 34



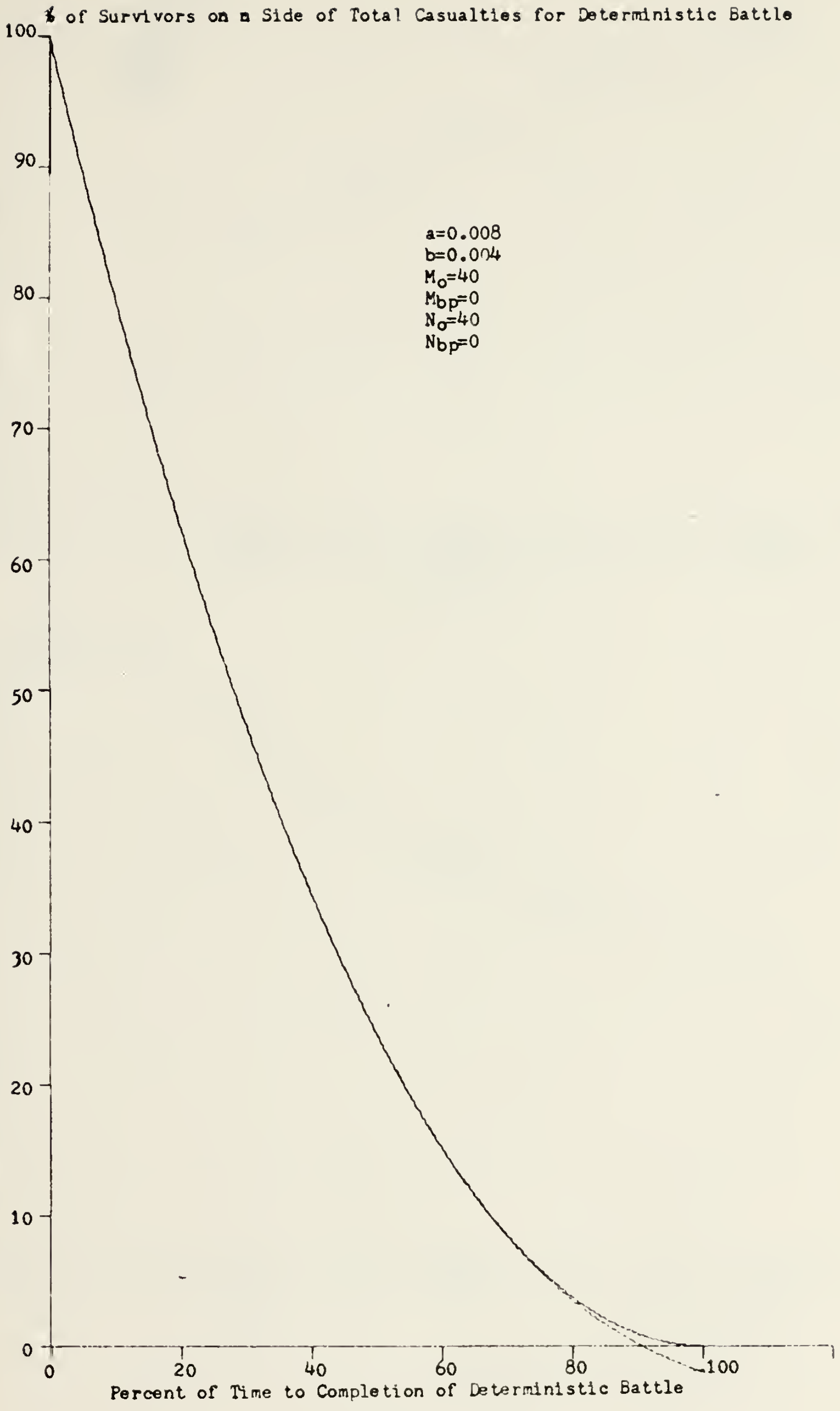
Time History Showing Bias

FIGURE 35



Time History Showing Bias

FIGURE 36



Time History Showing Bias

FIGURE 37

Δ_y and Δ_y for Increasing Initial Force Levels

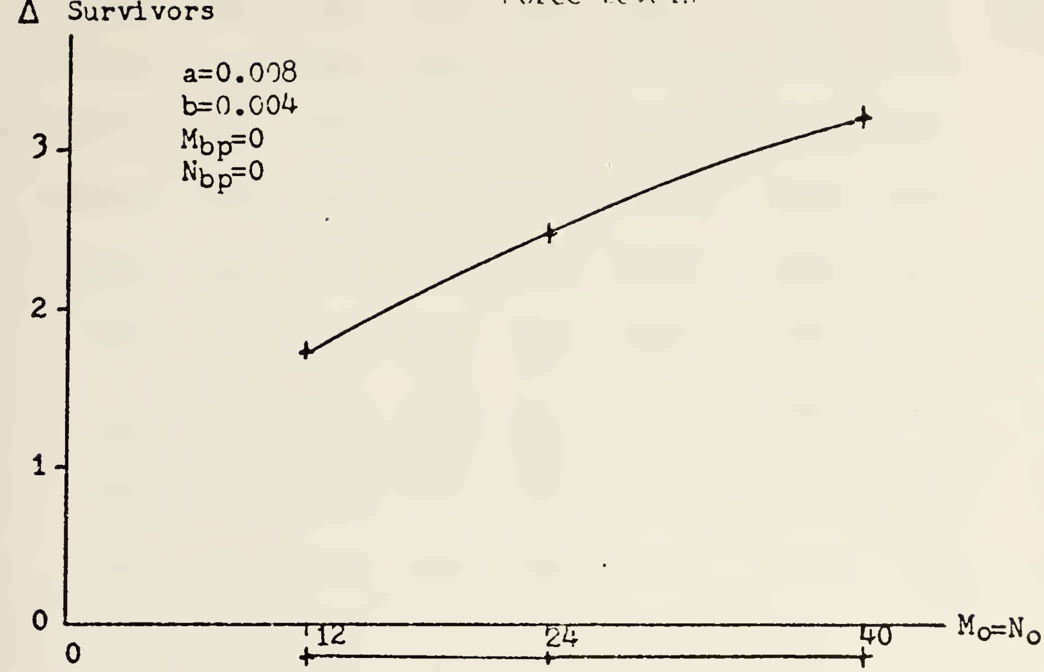
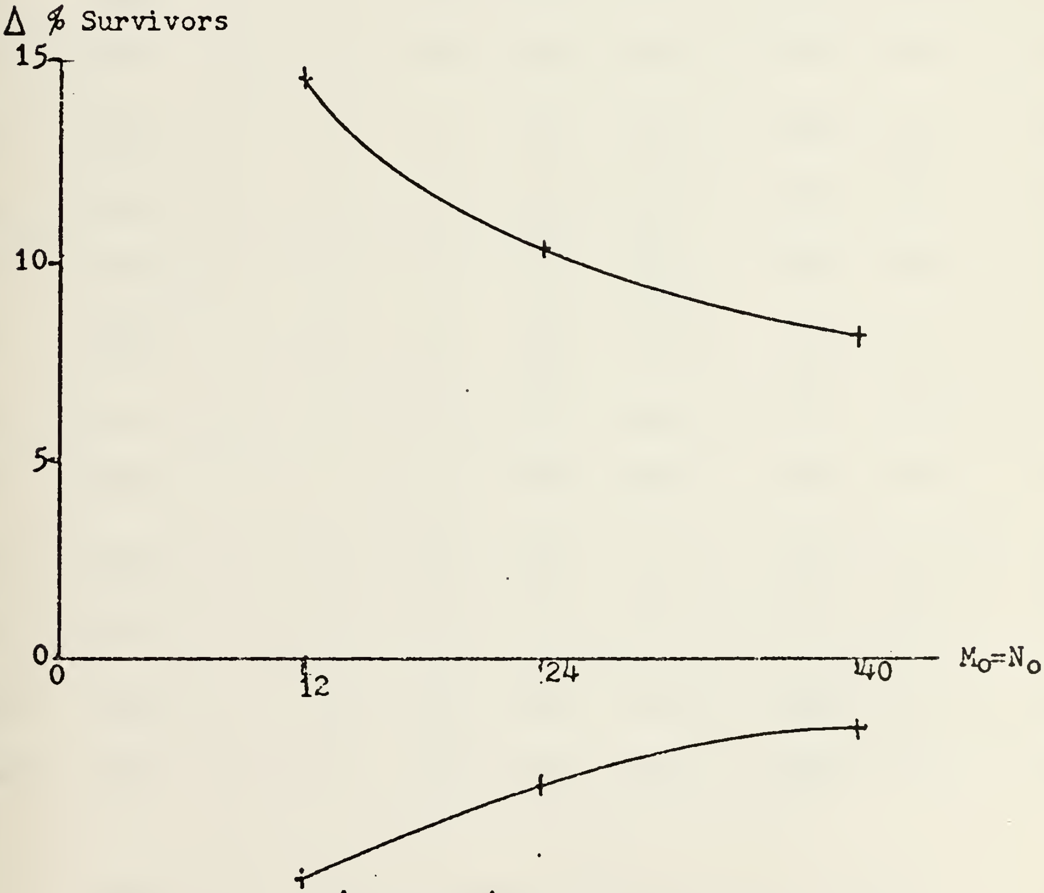


FIGURE 38



$\Delta\%_X$ and $\Delta\%_Y$ for Increasing Initial Force Levels

FIGURE 39

$\underline{a}$	$\underline{b}$	$\underline{F_X}$	$\underline{F_Y}$	$\underline{m_o}$	$\underline{n_o}$	$\underline{\Delta_X}$	$\underline{\Delta_X\%}$	$\underline{\Delta_Y}$	$\underline{\Delta_Y\%}$
0.008	0.004	1.0	1.0	12	12	1.75	14.58	-0.20	- 5.70
0.008	0.004	1.0	1.0	24	24	2.48	10.33	-0.22	- 3.13
0.008	0.004	1.0	1.0	40	40	3.22	8.05	-0.23	- 1.96
0.008	0.004	0.8	0.8	24	24	2.10	10.50	0.06	0.88
0.008	0.004	0.8	0.8	40	40	2.62	8.19	0.18	1.61
0.008	0.004	0.6	0.6	24	24	1.68	11.20	0.29	4.93
0.008	0.004	0.6	0.6	40	40	2.11	8.79	0.42	4.40
0.008	0.004	0.4	0.4	24	24	1.31	13.10	0.38	8.74
0.008	0.004	0.4	0.4	40	40	1.64	10.25	0.52	7.41
0.008	0.004	0.2	0.2	24	24	0.94	18.80	0.38	16.10
0.008	0.004	0.2	0.2	40	40	1.15	14.38	0.47	12.43
0.004	0.0015	1.0	1.0	6	6	1.13	18.83	-0.10	- 7.94
0.004	0.0015	1.0	1.0	8	8	1.31	16.38	-0.11	- 6.55
0.004	0.0015	1.0	1.0	12	12	1.61	13.42	-0.12	- 4.78
0.004	0.0015	1.0	1.0	24	24	2.28	9.50	-0.13	- 2.58
0.004	0.0015	0.6	0.6	24	24	1.63	10.87	0.19	4.48
0.004	0.0015	0.6	0.6	40	40	2.06	8.58	0.28	4.06
0.004	0.0015	0.4	0.4	24	24	1.31	13.10	0.38	8.74
0.004	0.0015	0.4	0.4	40	40	1.64	10.25	0.52	7.41
0.004	0.0015	0.2	0.2	24	24	0.94	18.80	0.38	16.10
0.004	0.0015	0.2	0.2	40	40	1.15	14.38	0.47	12.43

TABLE III. Bias for Different Initial Force Levels

The general result of this hypothesis is that, if the forces are of any appreciable size (greater than 20), with breakpoint force levels such that $m_o - m_{bp}$ and $n_o - n_{bp}$ both exceed 20, the deterministic model is probably an adequate representation of the complex random process of combat.

Hypothesis 2-3

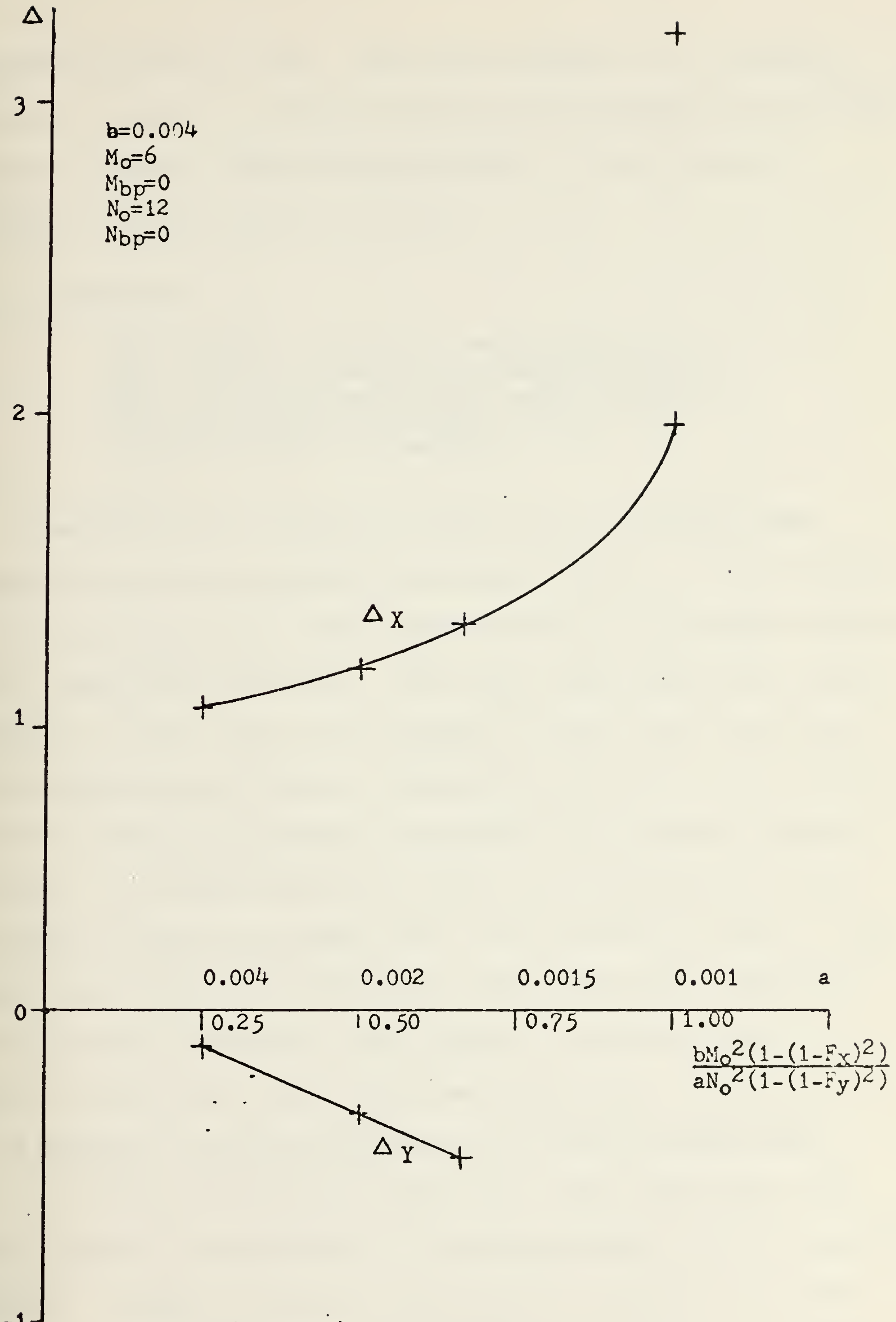
> If any parameter is varied to bring the forces closer to parity, the time of the battle increases and the bias increases.

This hypothesis is a result of an analysis of the solution for $\Delta_X(t)$ and $\Delta_Y(t)$. As was shown previously,

$$\Delta_X(t) = \int_0^t [S_Y(\tau)\cosh(\sqrt{ab}(t-\tau)) - S_X(\tau)\sqrt{a/b}\sinh(\sqrt{ab}(t-\tau)]d\tau \tag{60}$$

$$\Delta_Y(t) = \int_0^t [S_X(\tau)\cosh(\sqrt{ab}(t-\tau)) - S_Y(\tau)\sqrt{a/b}\sinh(\sqrt{ab}(t-\tau)]d\tau \tag{61}$$

It appears that, as t increases, the integral will increase in absolute value. As only one parameter is varied, numerical parametric analysis was easy. All results computed indicated the hypothesis is true. Figure 40 shows typical results. It is interesting to note that the hypothesis holds for the loser. However, it is not clear what happens for the winner, but it would appear that it is a continuous function. If it is, then at some point, Δ_Y reaches a



Δ_X and Δ_Y Versus Winning Criteria

FIGURE 40

maximum negative value, and then increases to a positive value at parity. It then appears the hypothesis only holds for the loser. The results in Table IV show the same trends for the winner and loser.

Hypothesis 2-4

> If a battle were to be terminated at a time $t \leq 0.5t_f$, where t_f is the completion time of the deterministic model, there is no significant difference between the results of the two models.

Decisions to terminate battles are not always made because too many casualties have been taken. A good example of this would be the rear guard action of a division involved in a retrograde movement. The rear guard delays the enemy force, in many cases, long enough to allow the main force to retrograde and set up a defensive position, or at least allow a fresh rear guard to prepare for a further delaying action. In either case the rear guard commander will be tasked to delay the enemy for a fixed amount of time. When that time limit is reached, he will disengage, regardless of whether he feels he could engage the enemy longer. In fact, he may have taken very few casualties. This would be a situation in which hypothesis 2-4 would be applicable.

An evaluation of $\Delta_X(t)$ and $\Delta_Y(t)$ will show that they are generally increasing (in absolute terms), with increasing time. A study of various combinations of parameters has shown this to be the case. The question then is, at what

$\underline{a}$	$\underline{b}$	$\underline{m_o}$	$\underline{m_{bp}}$	$\underline{n_o}$	$\underline{n_{bp}}$	$\underline{\Delta_X}$	$\underline{\Delta_Y}$	$\underline{t_f}$	$\underline{\dfrac{bm_o^2(1-(1-F_X)^2)}{an_o^2(1-(1-F_Y)^2)}}$
0.008	0.004	24	19	24	0	0.88	0.35	27	0.18
0.008	0.004	24	19	24	0	1.30	0.37	58	0.32
0.008	0.004	24	9	24	0	1.68	0.29	91	0.42
0.008	0.004	24	4	24	0	2.10	0.06	126	0.48
0.008	0.004	24	0	24	0	2.48	-0.22	156	0.50
0.004	0.001	6	0	6	0	1.06	-0.07	275	0.25
0.004	0.001	12	0	6	0	3.22	1.93	493	1.00
0.004	0.001	6	0	6	0	1.06	-0.07	275	0.25
0.004	0.0015	6	0	6	0	1.13	-0.10	291	0.375
0.004	0.002	6	0	6	0	1.23	-0.14	312	0.50
0.004	0.004	6	0	6	0	1.93	1.93	∞	1.00
0.004	0.004	6	0	12	0	1.06	-0.11	137	0.25
0.002	0.004	6	0	12	0	1.18	-0.32	312	0.50
0.0015	0.004	6	0	12	0	1.34	-0.47	493	0.75
0.001	0.004	6	0	12	0	1.97	3.22	∞	1.00

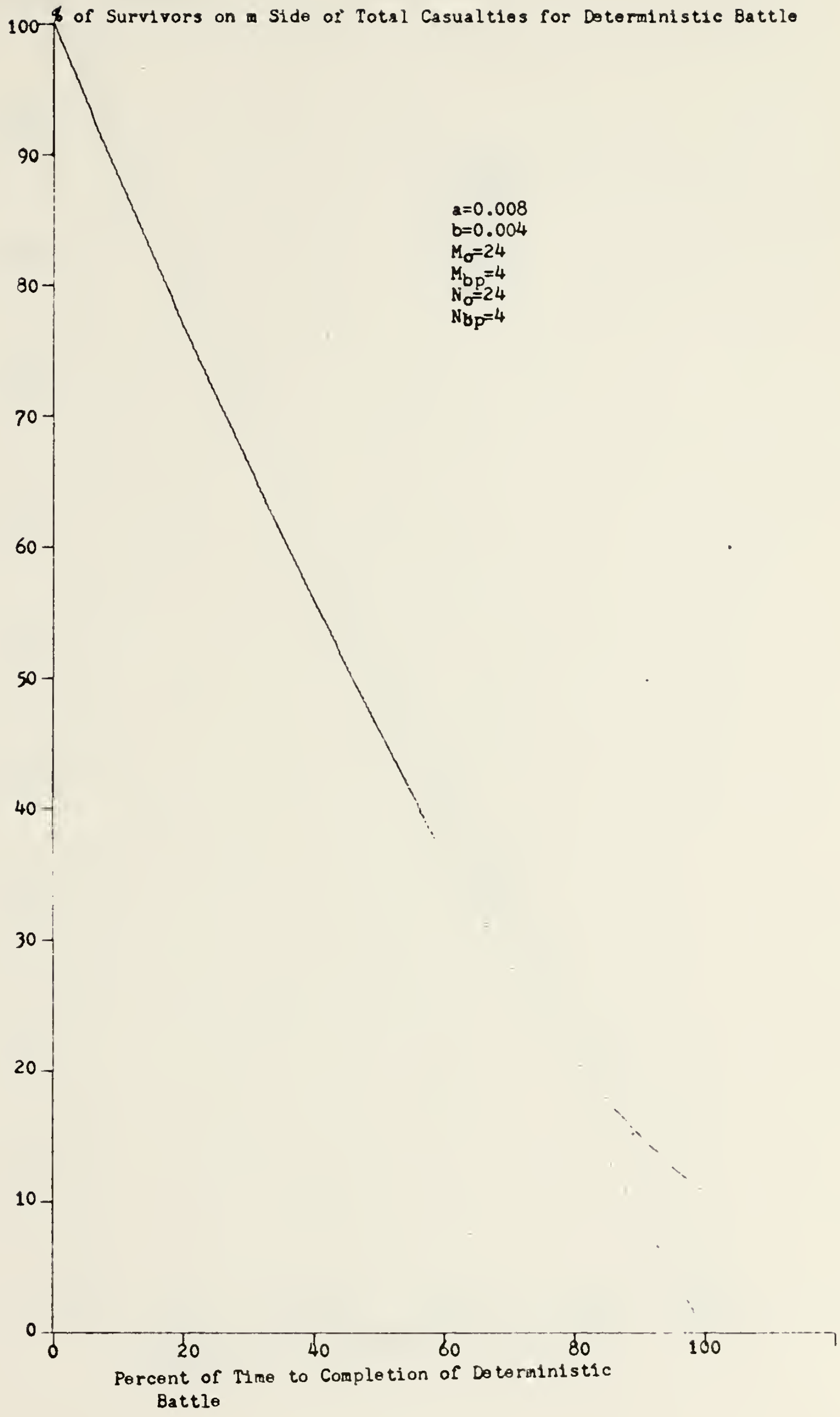
TABLE IV. Bias for Different Attrition Rate Coefficients

time into the battle do $\Delta_X(t)$ and $\Delta_Y(t)$ become significant? No method was found to determine this, but Figures 41 to 69 are typical of all results found.

As might be expected, the time at which the two time histories begin to diverge is strongly related to the final differences $\Delta_X\%(t_f)$ and $\Delta_Y\%(t_f)$. As has been noted, as $m_o - m_{bp}$ and $n_o - n_{bp}$ decrease, and the force levels are moved closer to parity, $\Delta_X\%(t_f)$ and $\Delta_Y\%(t_f)$ increase. The worst examples found did not diverge before 0.5 t_f, with one excaption. The one exception to this hypothesis is when the forces are at or very near parity. In this case, the hypothesis does not hold because, with the forces near or at parity, the rate at which casualties on both sides occur approaches zero as time increases. This gives, in a sense, an exaggerated time length of the battle and thereby distorts the relative amount of time $\Delta_X\%$ and $\Delta_Y\%$ are appreciable.

C. VARIANCE

It is possible for the two models to agree on which side is going to win, and on the time histories of the average force levels, but the deterministic model may not adequately represent the complex random process of combat. This is because the outcome of the stochastic model is determined by the outcome of many uncertain events. The probability distribution function related to these uncertainties may cause the variance of the force levels to be considerable.



Time History Showing Bias

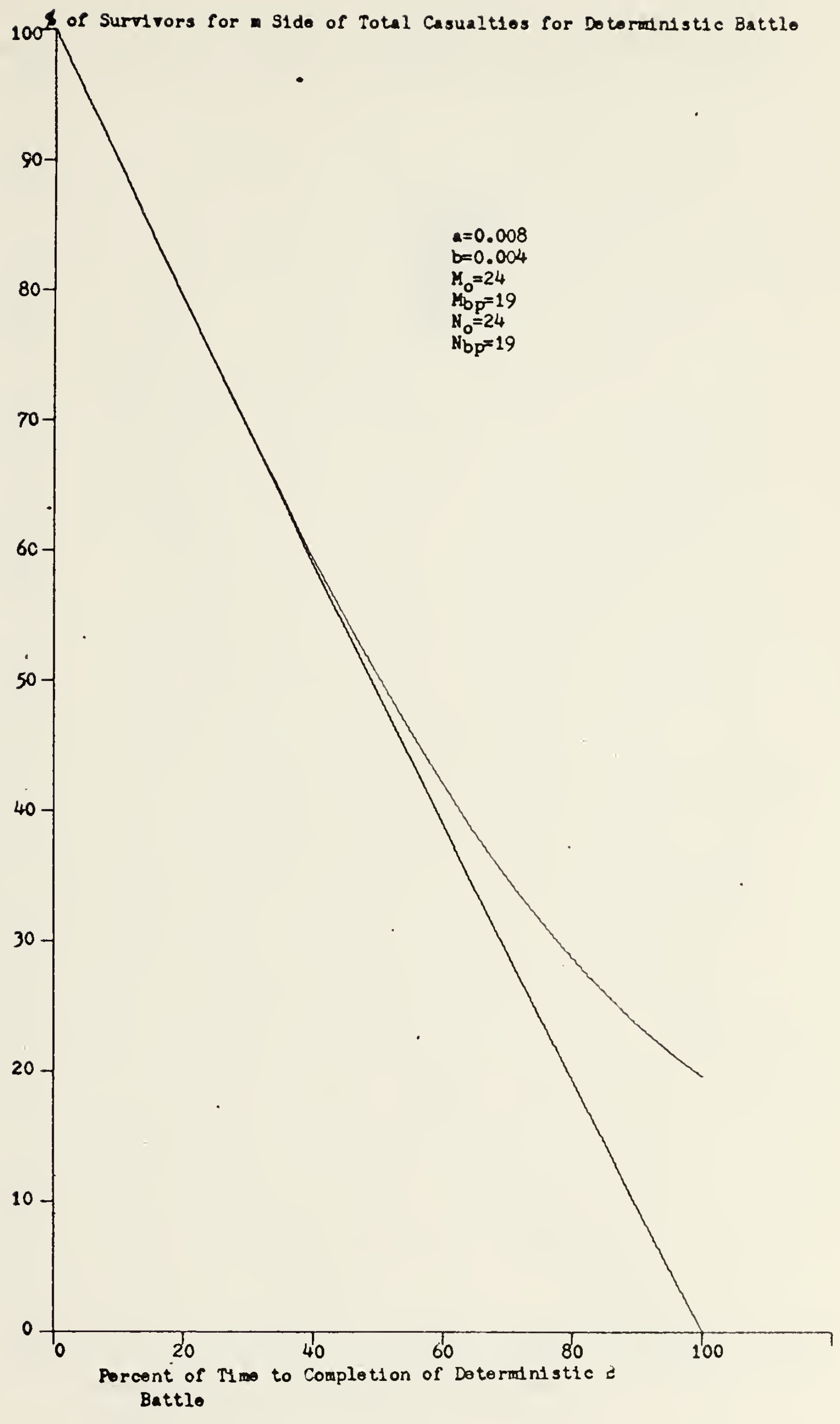
FIGURE 41

Time History Showing Bias

FIGURE 42

Time History Showing Bias

FIGURE 43



Time History Showing Bias

FIGURE 44

Time History Showing Bias

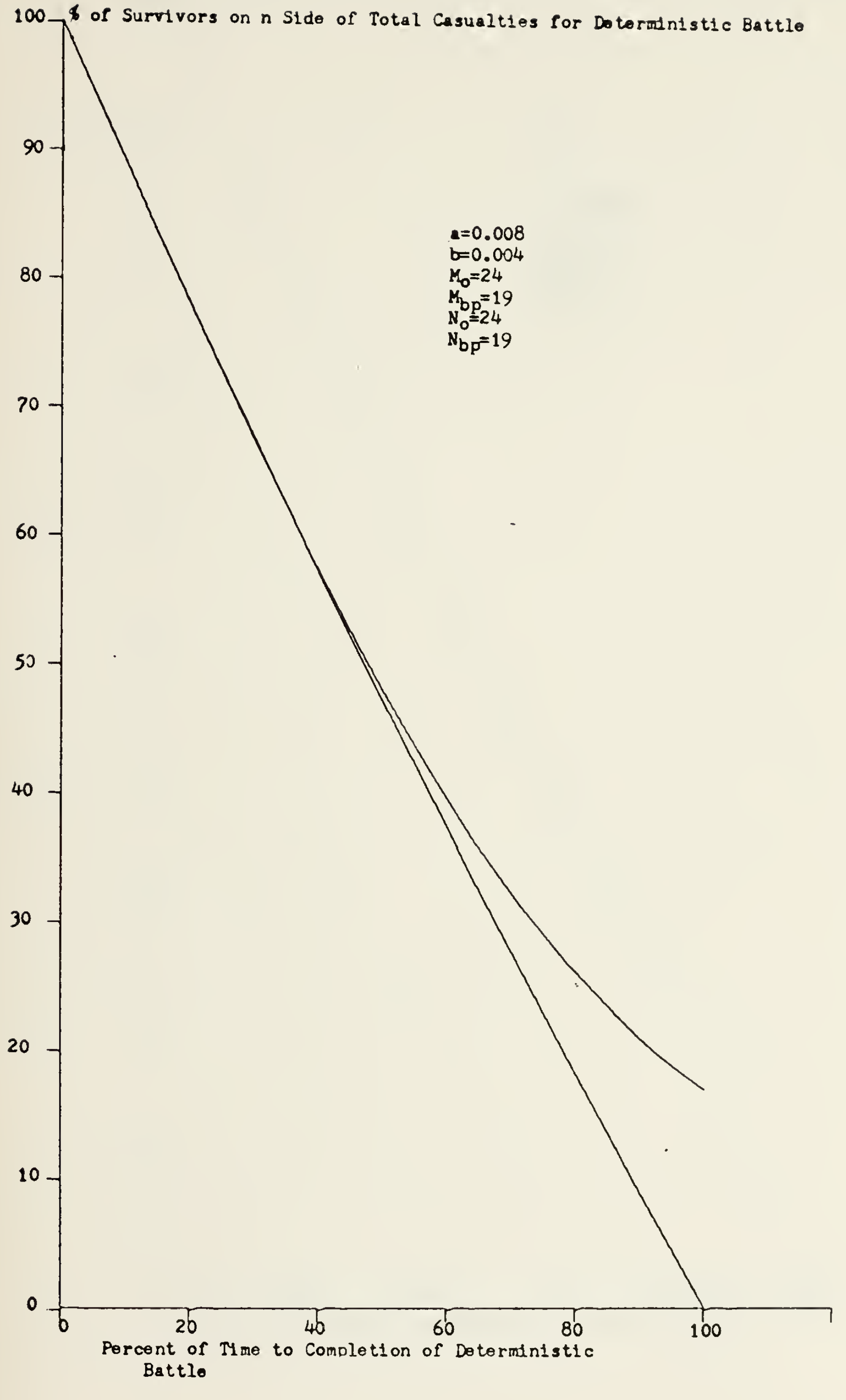
FIGURE 45

Time History Showing Bias

FIGURE 46

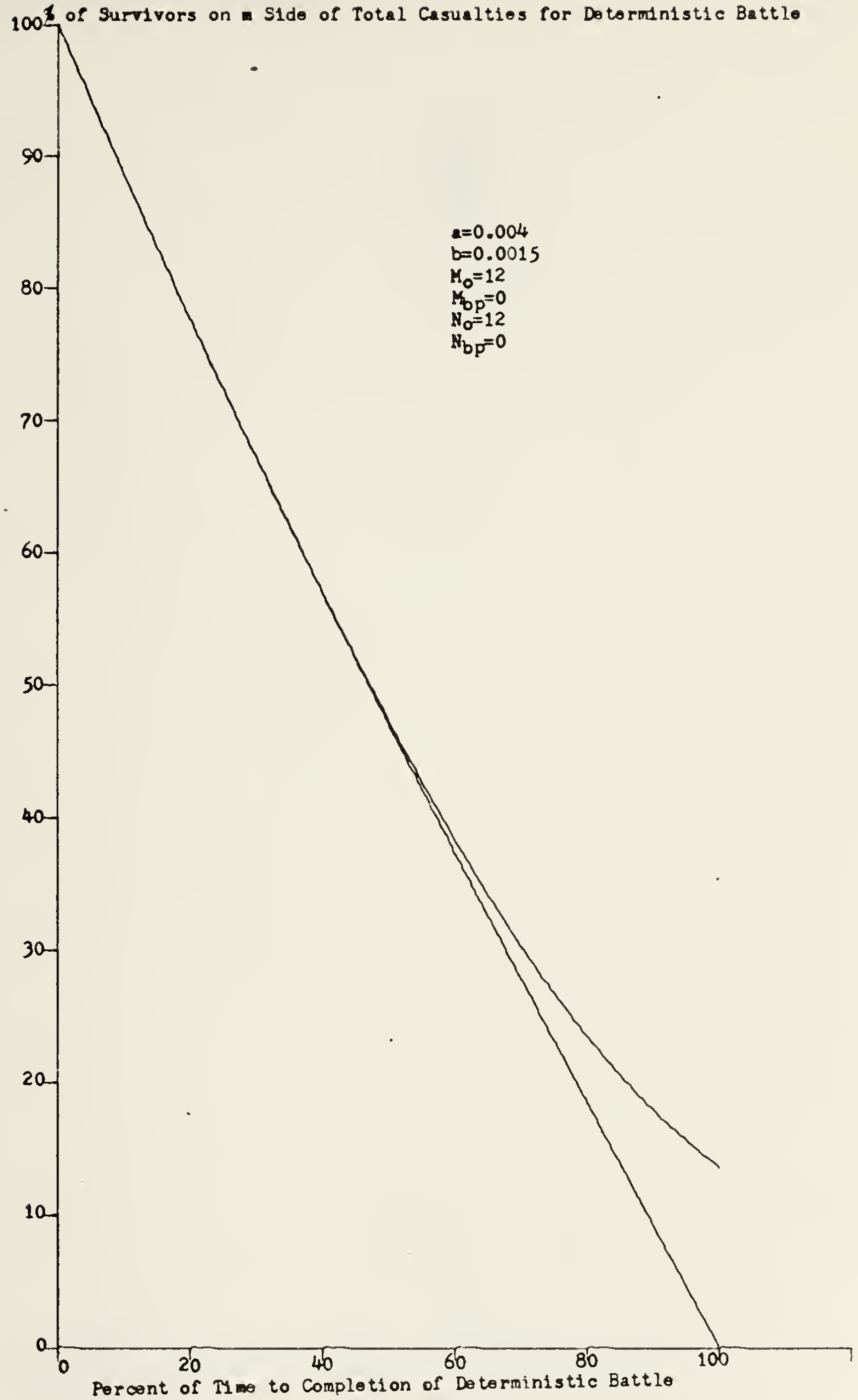
Time History Showing Bias

FIGURE 47



Time History Showing Bias

FIGURE 48



Time History Showing Bias

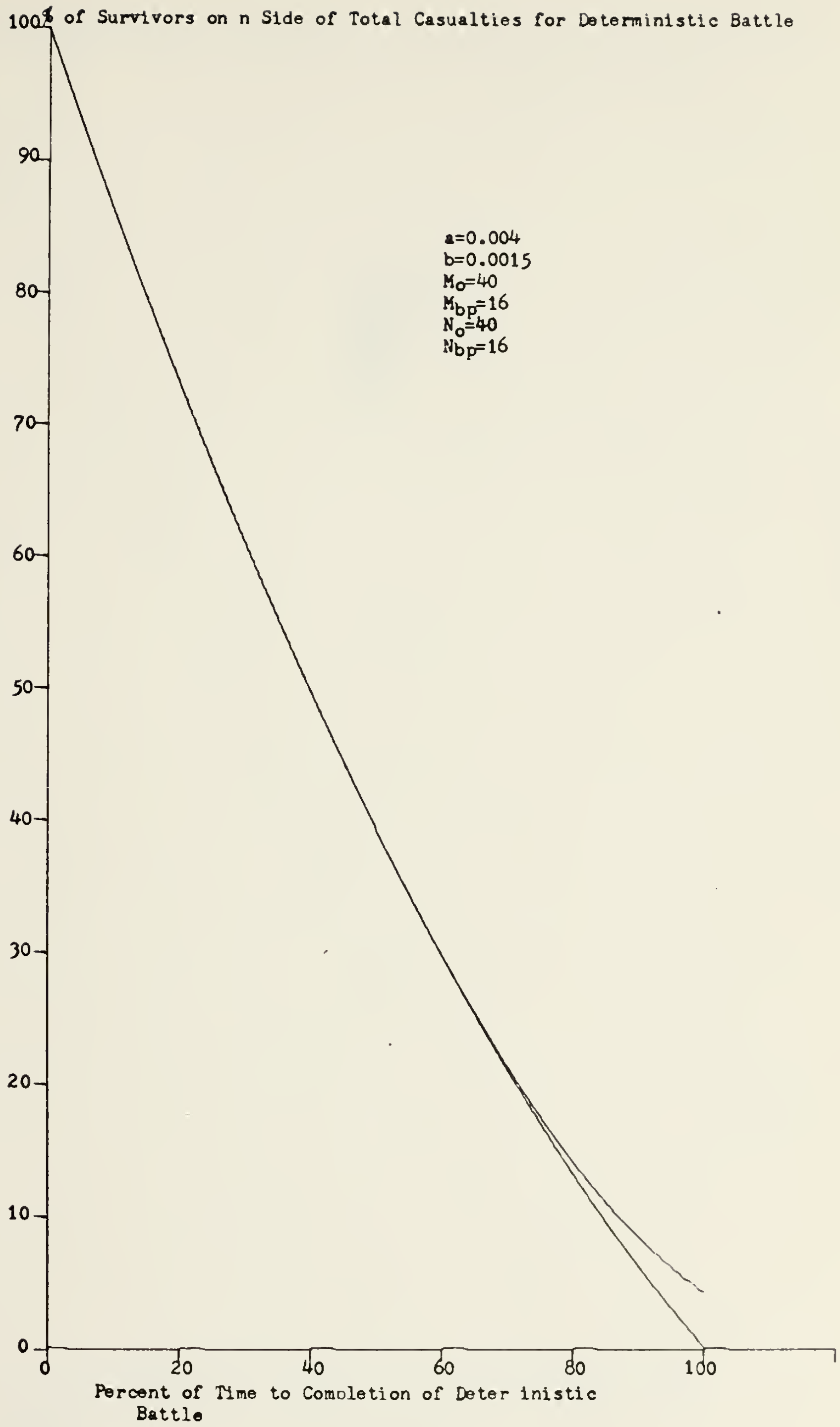
FIGURE 49

Time History Showing Bias

FIGURE 50

Time History Showing Bias

FIGURE 51

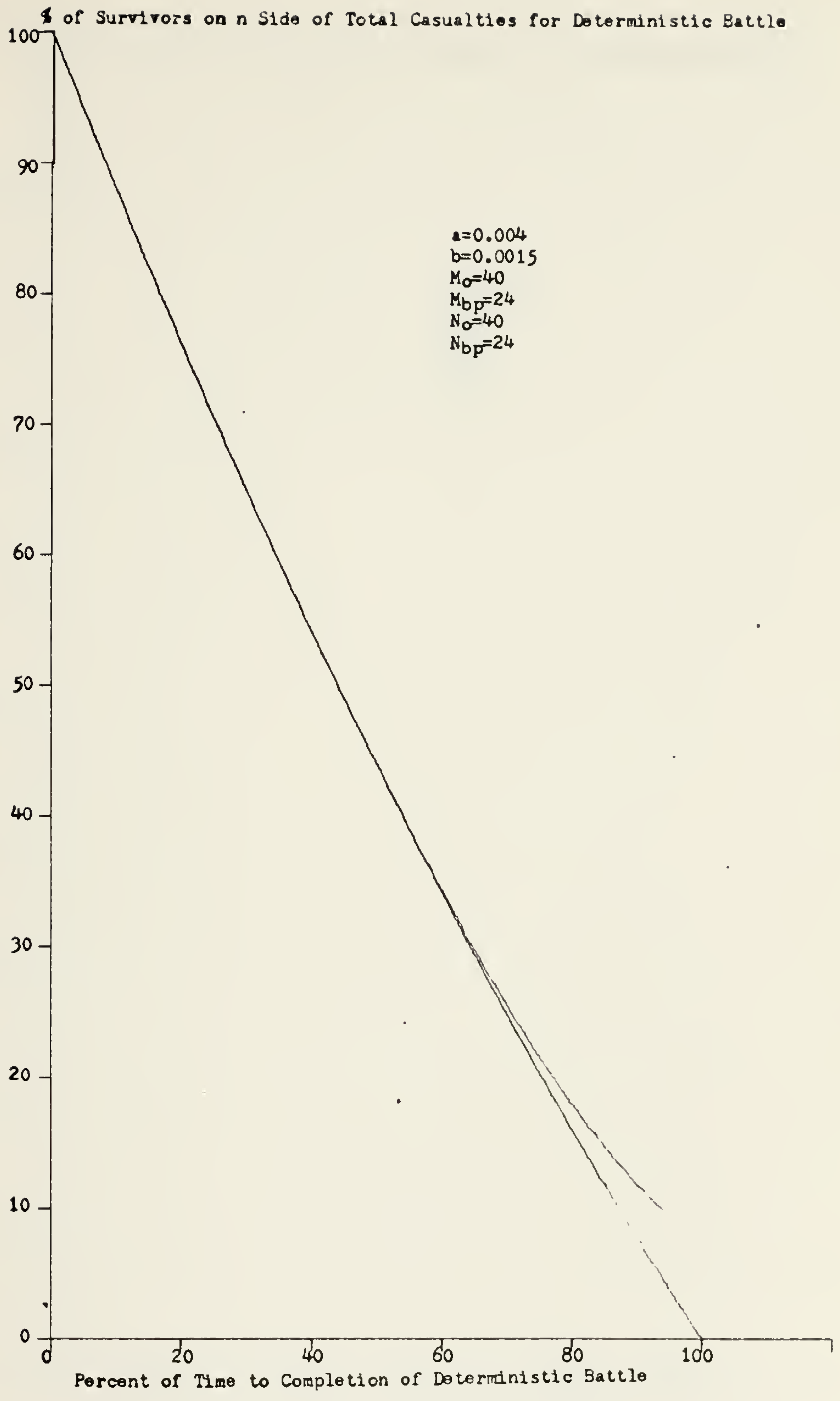


Time History Showing Bias

FIGURE 52

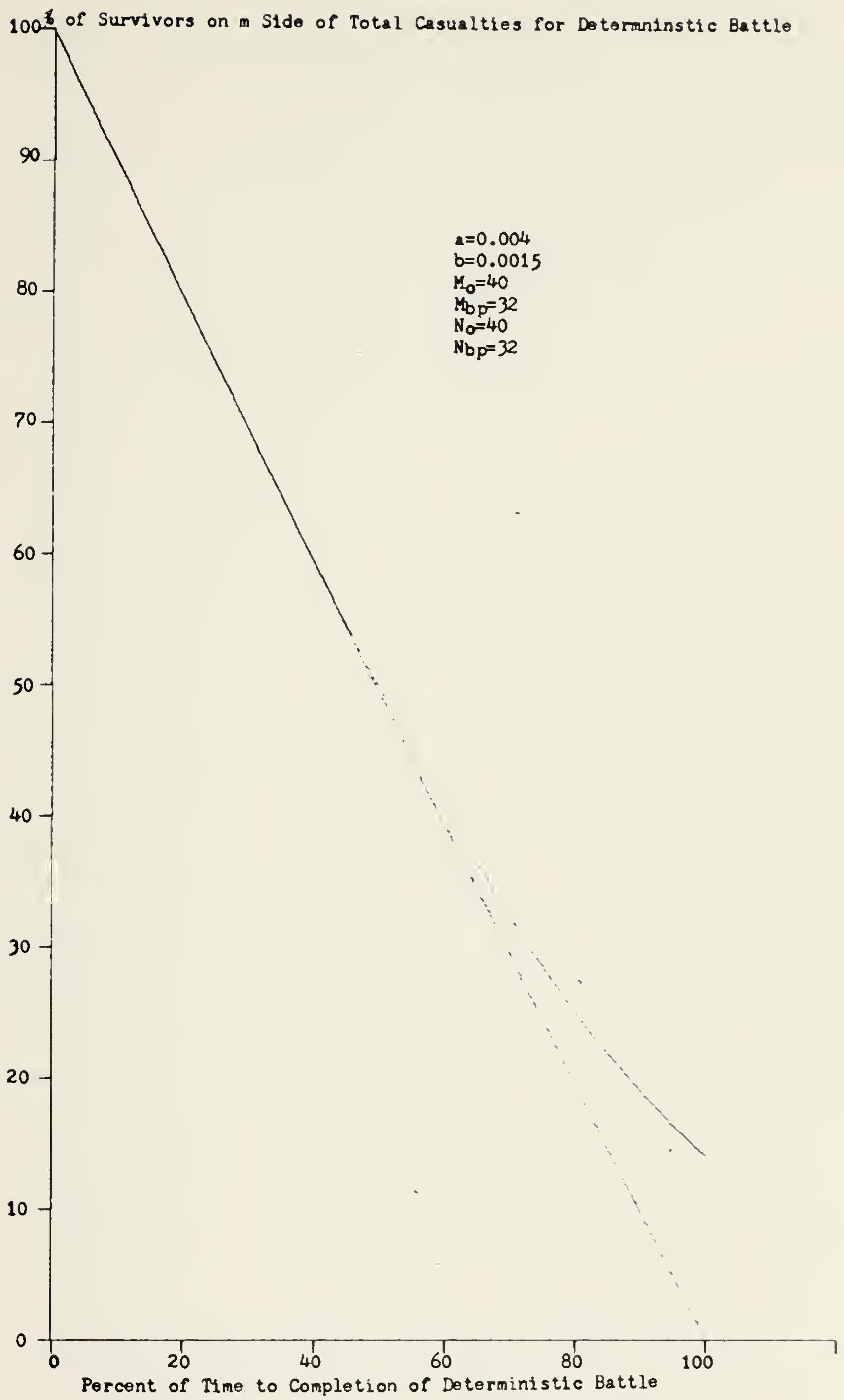
Time History Showing Bias

FIGURE 53



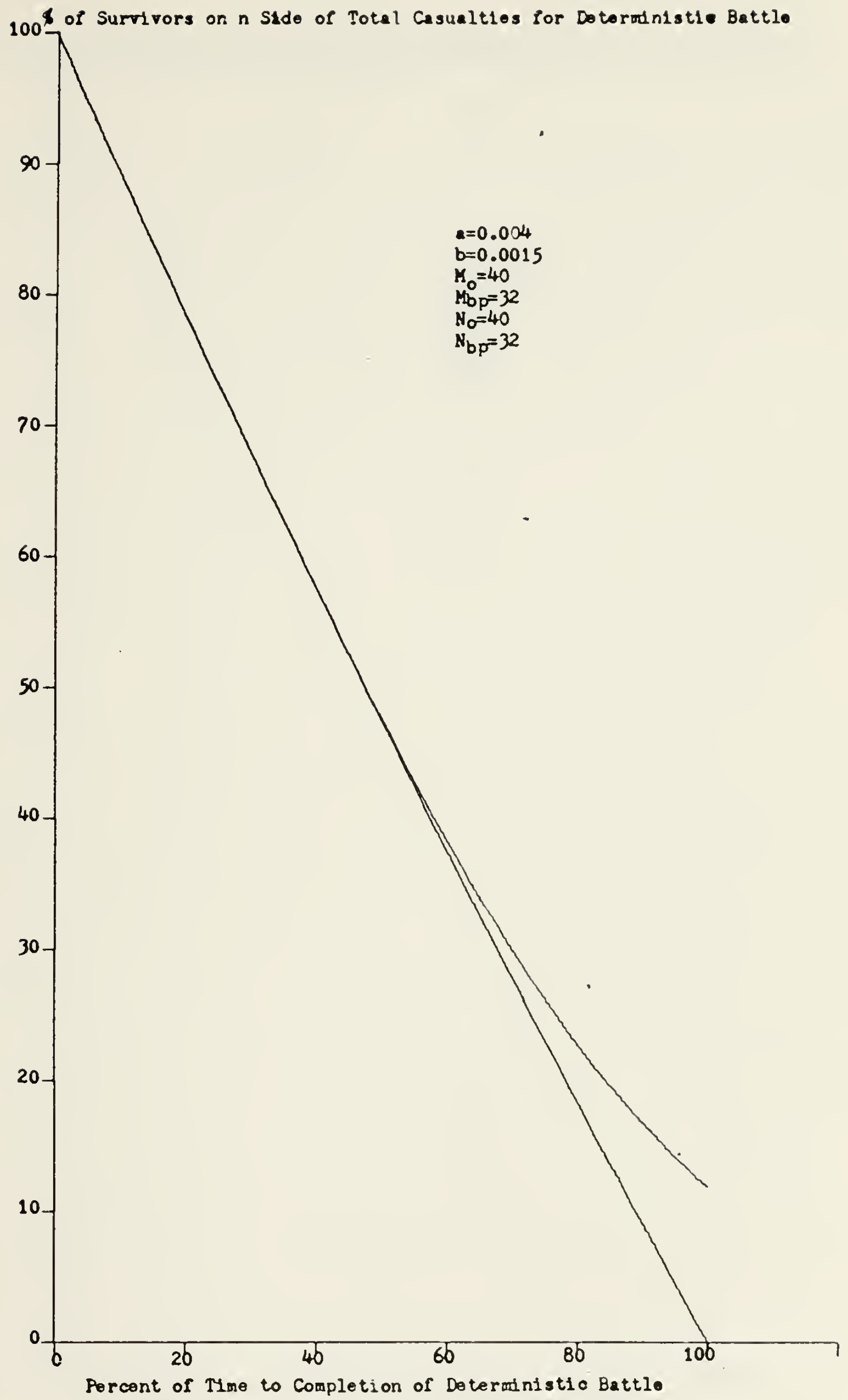
Time History Showing Bias

FIGURE 54



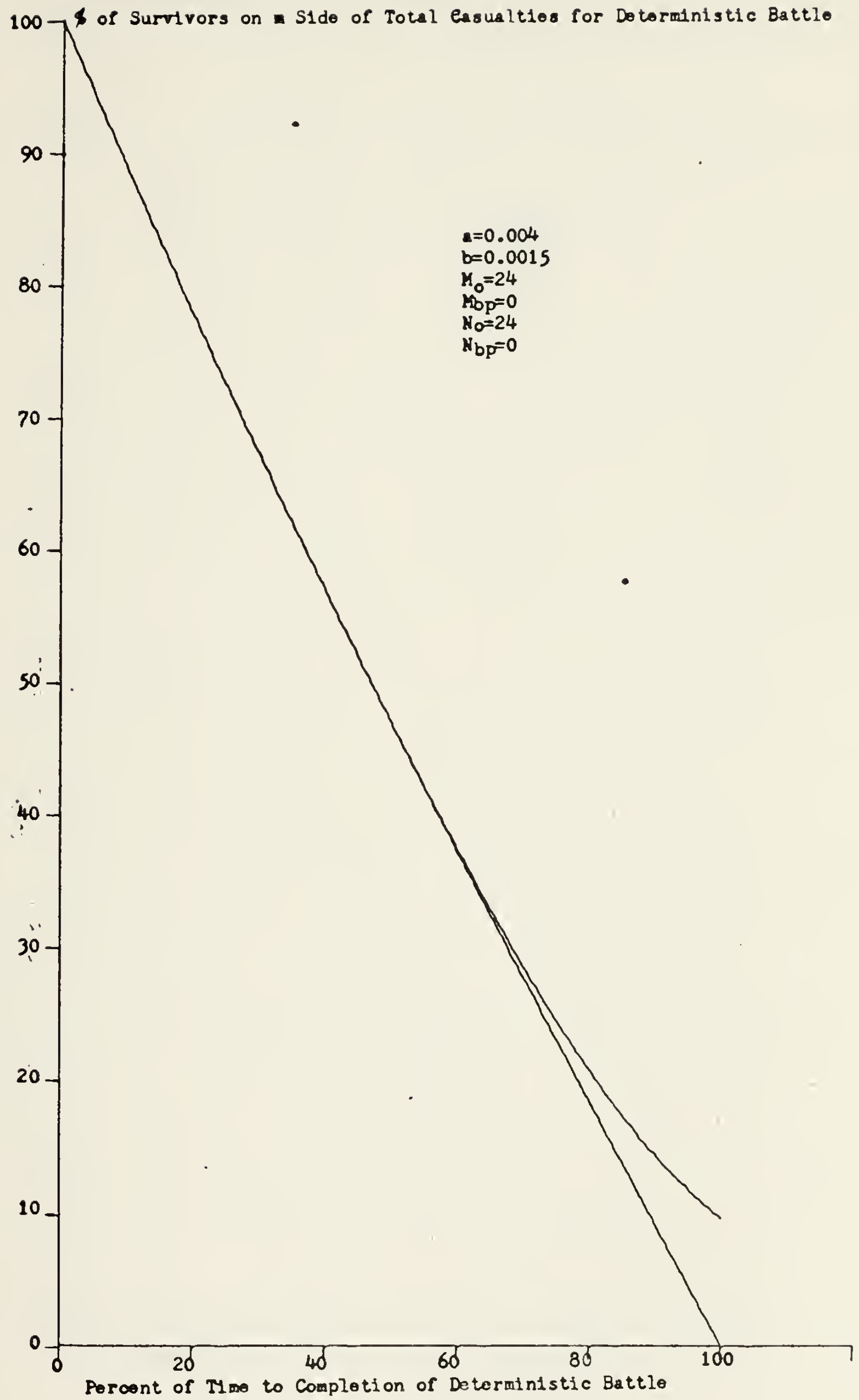
Time History Showing Bias

FIGURE 55



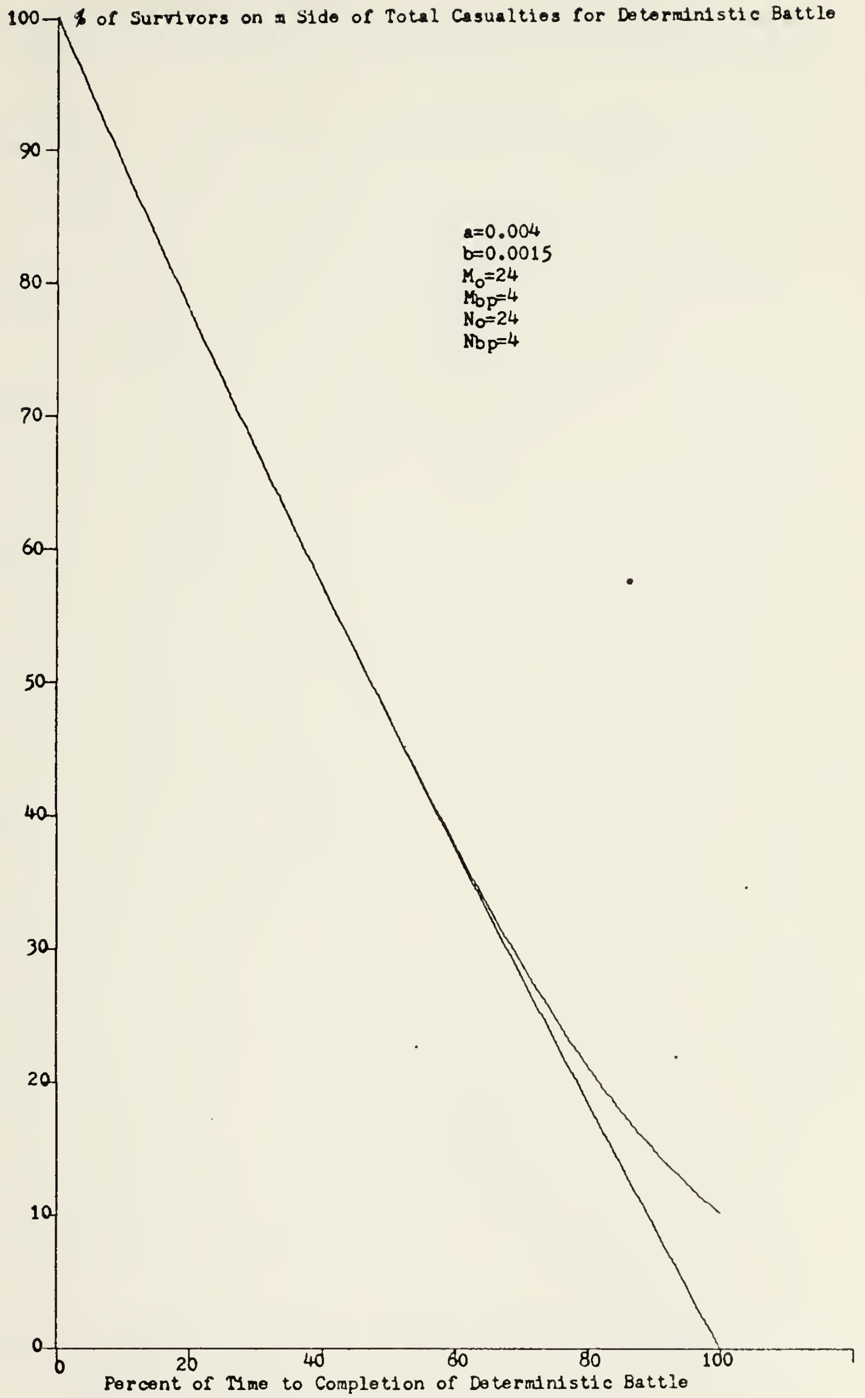
Time History Showing Bias

FIGURE 56



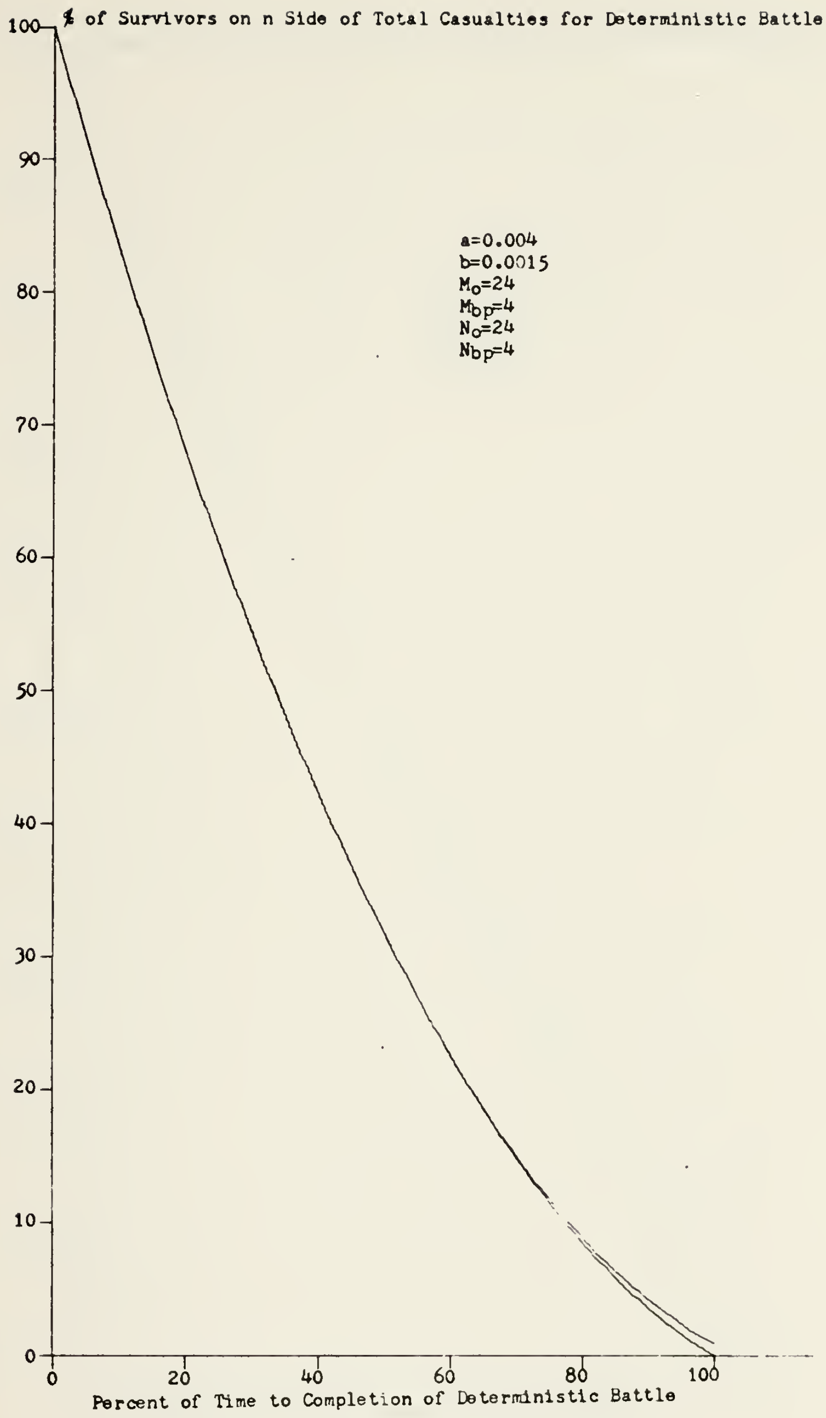
Time History Showing Bias

FIGURE 57



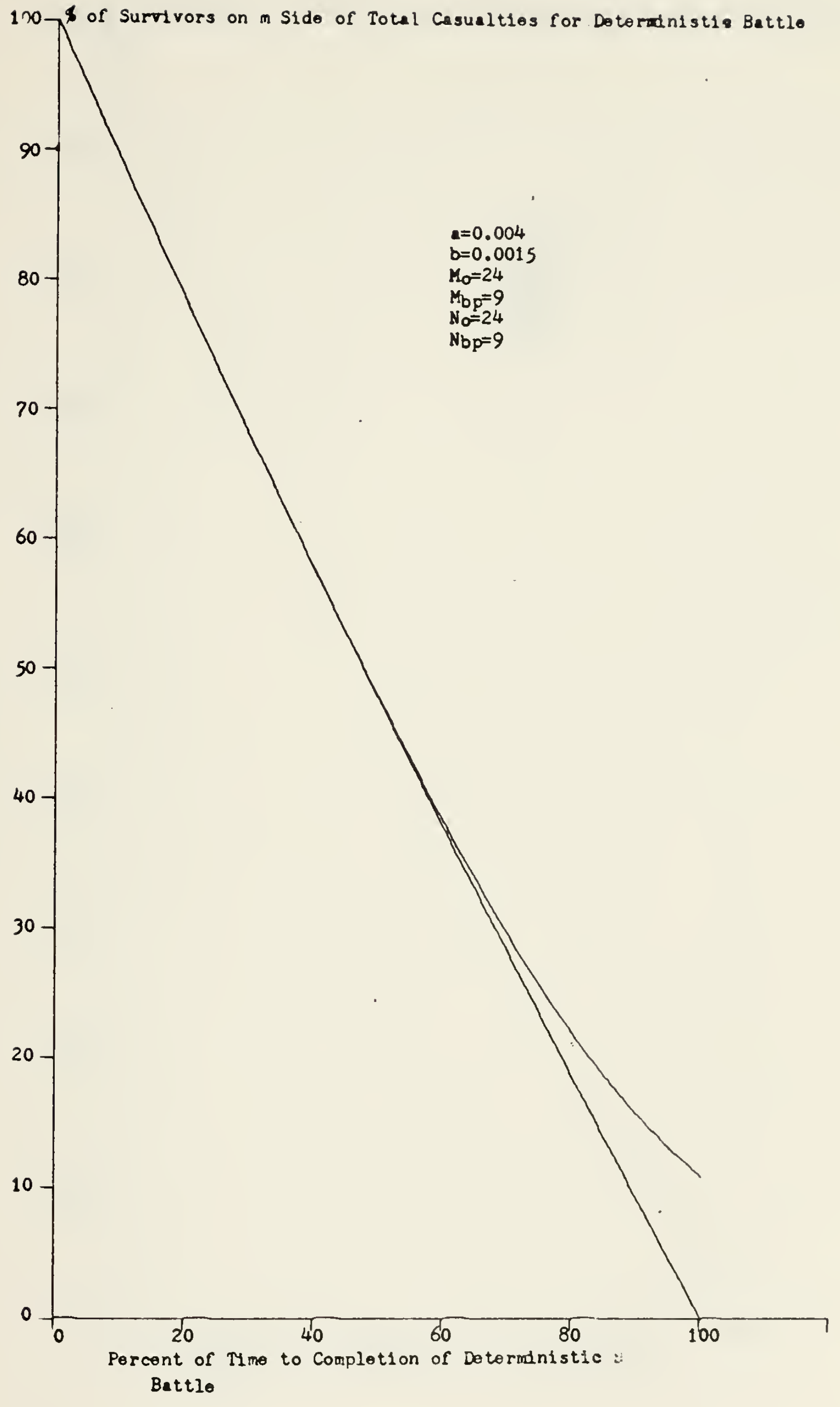
Time History Showing Bias

FIGURE 58



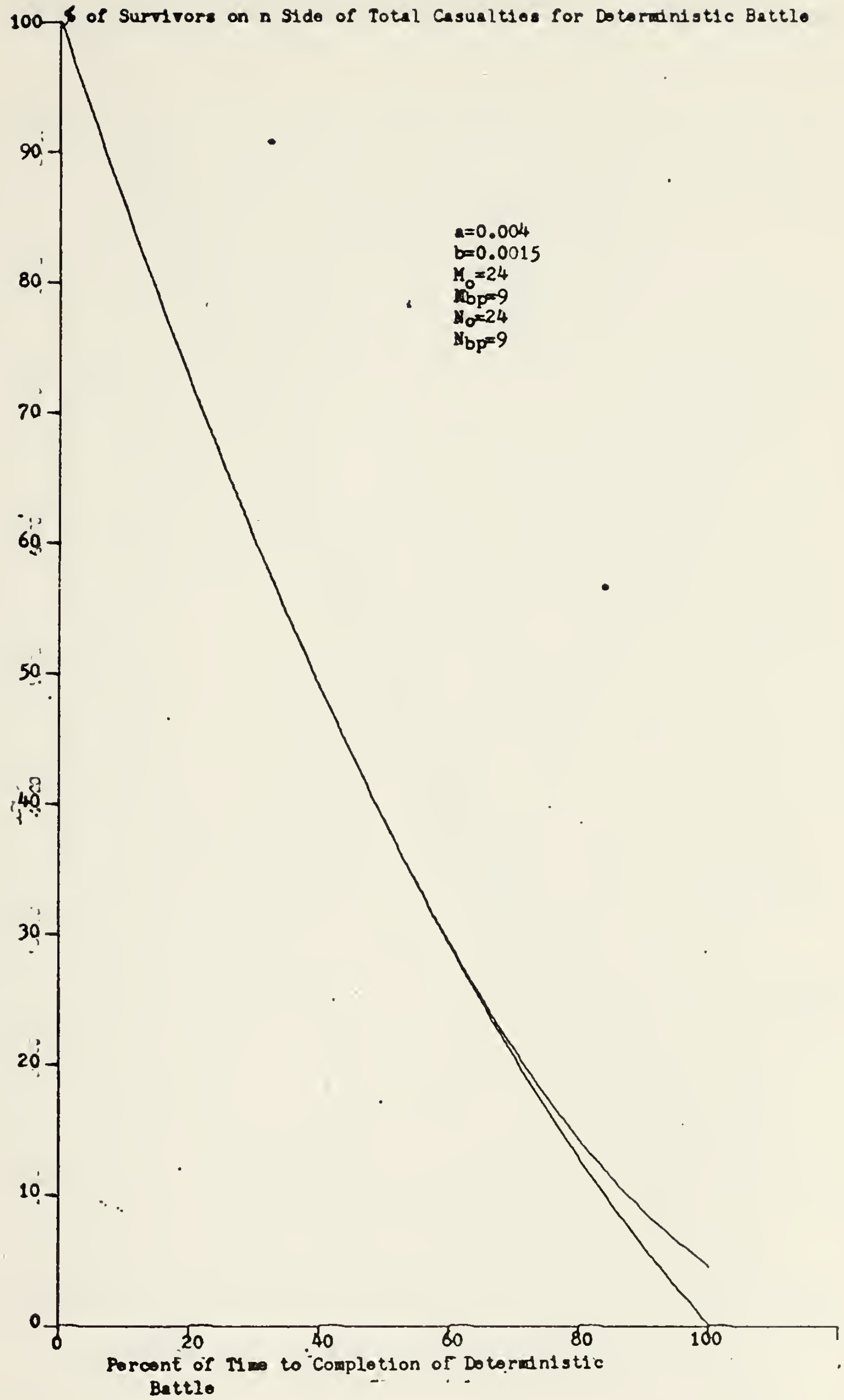
Time History Showing Bias

FIGURE 59



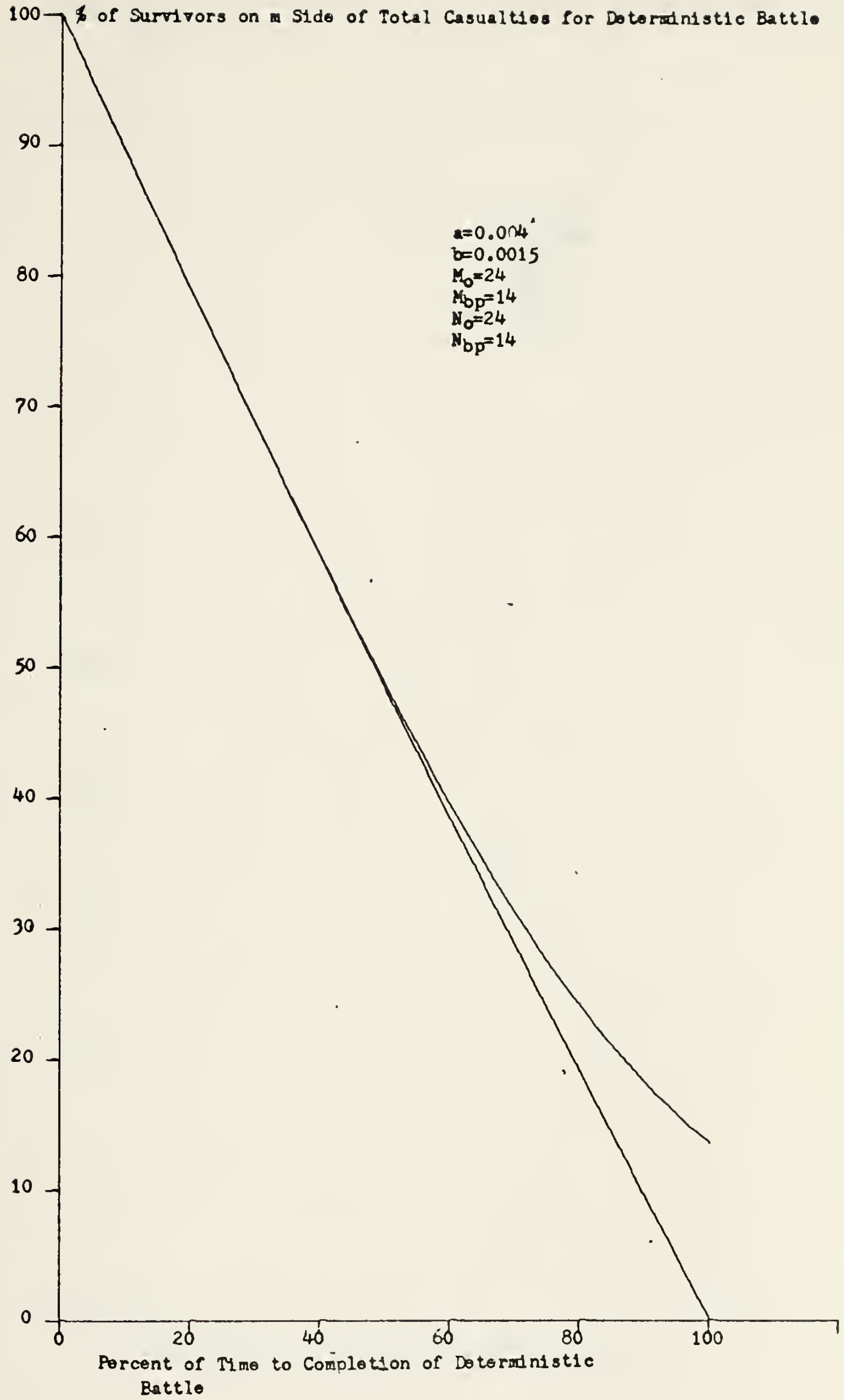
Time History Showing Bias

FIGURE 60



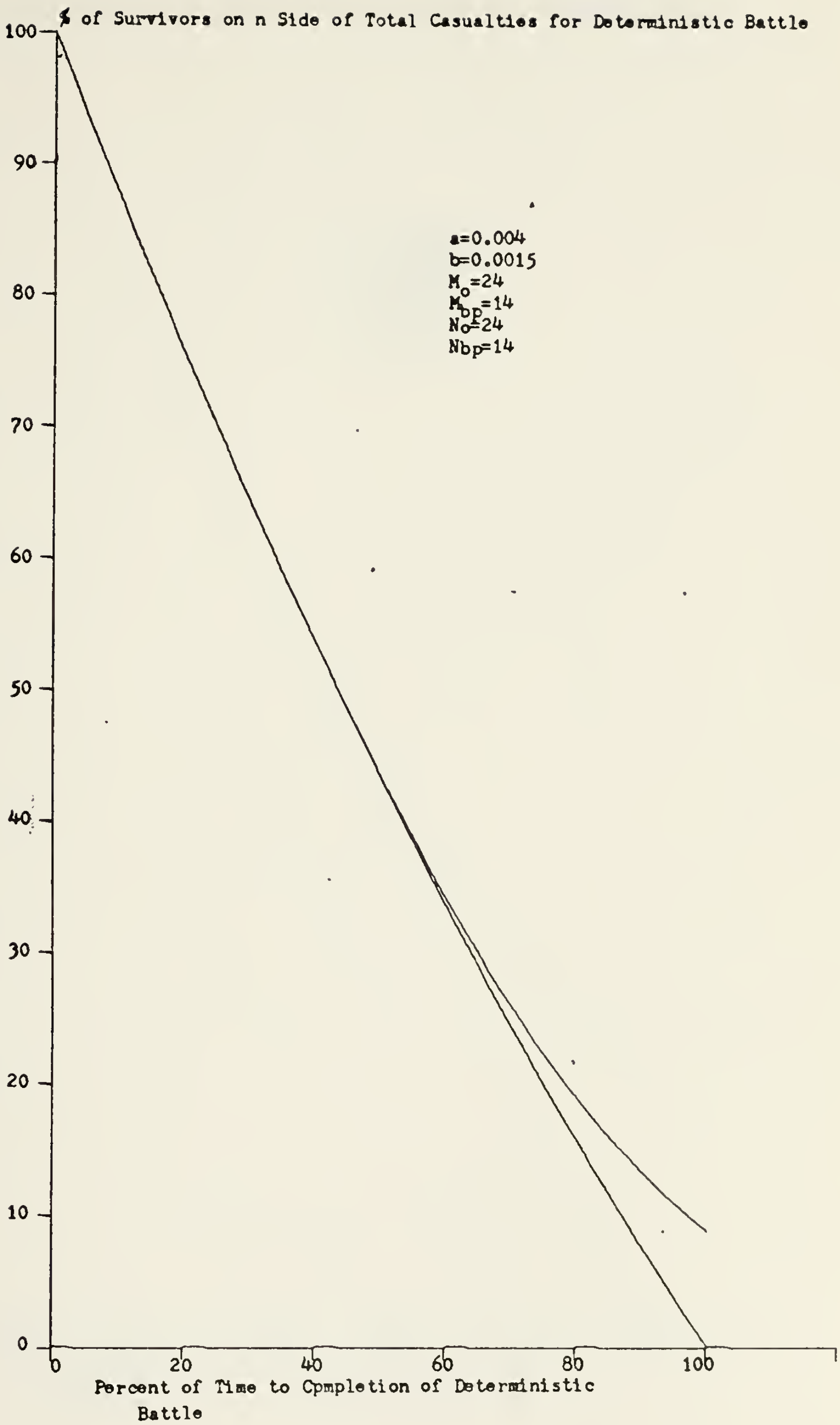
Time History Showing Bias

FIGURE 61



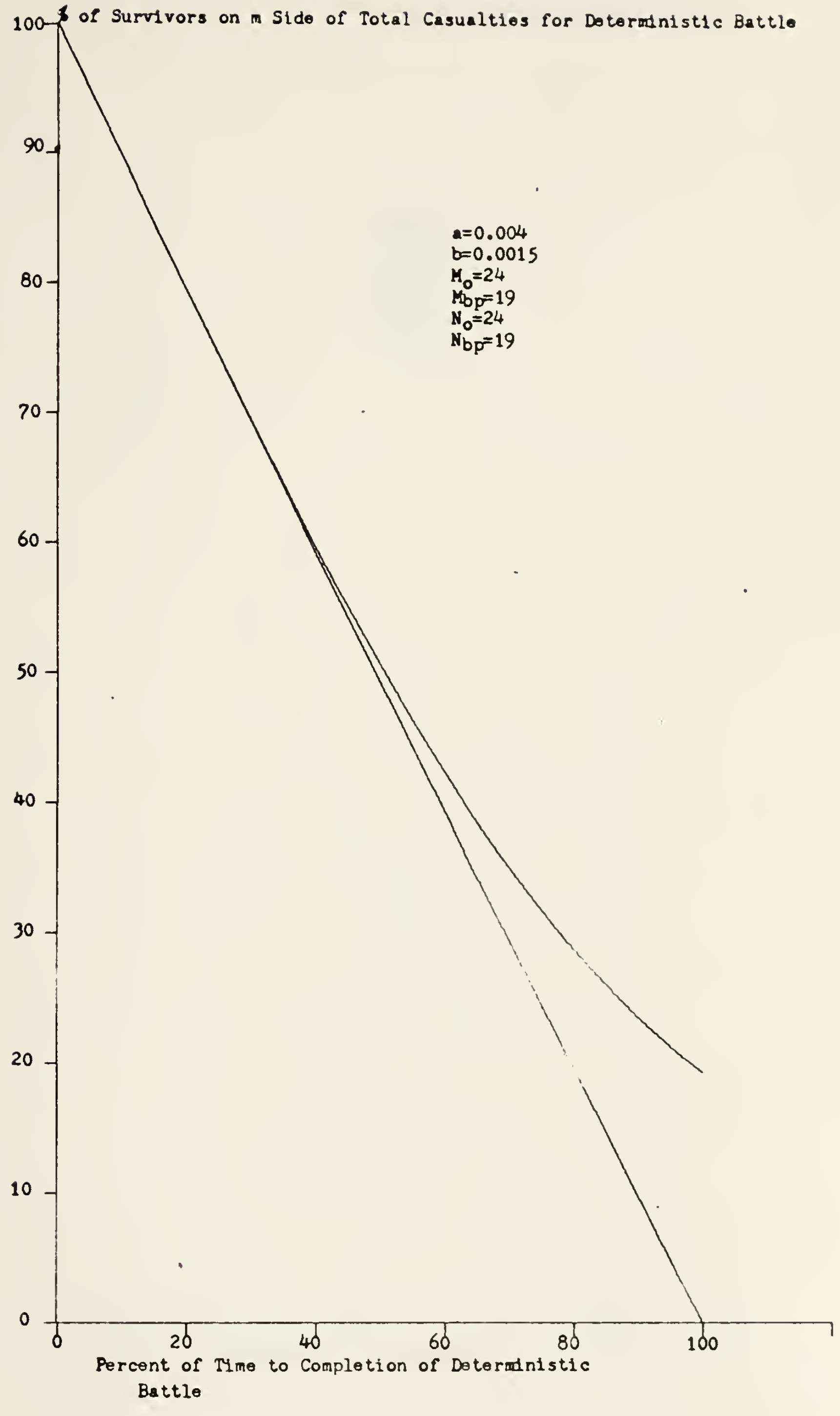
Time History Showing Bias

FIGURE 62



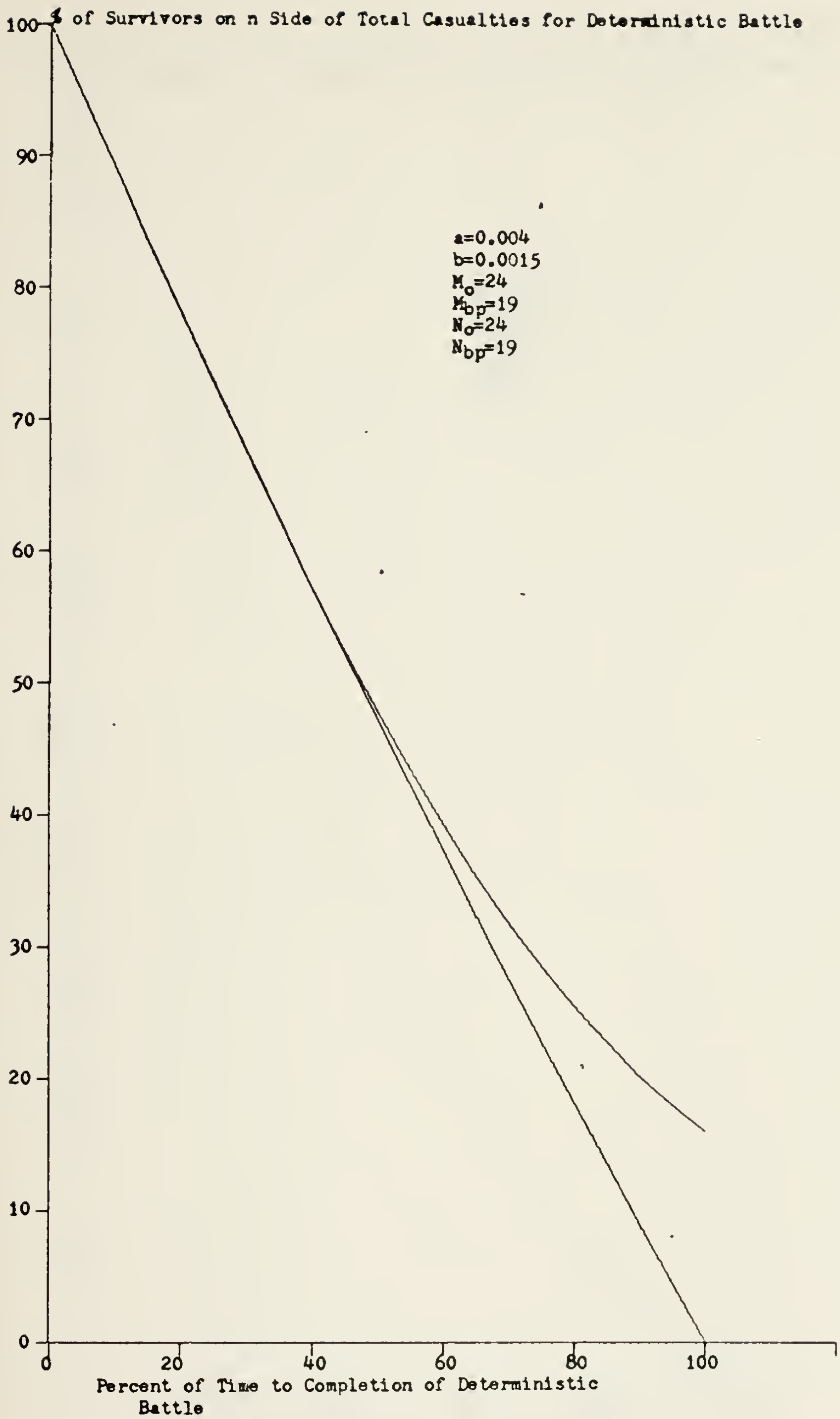
Time History Showing Bias

FIGURE 63



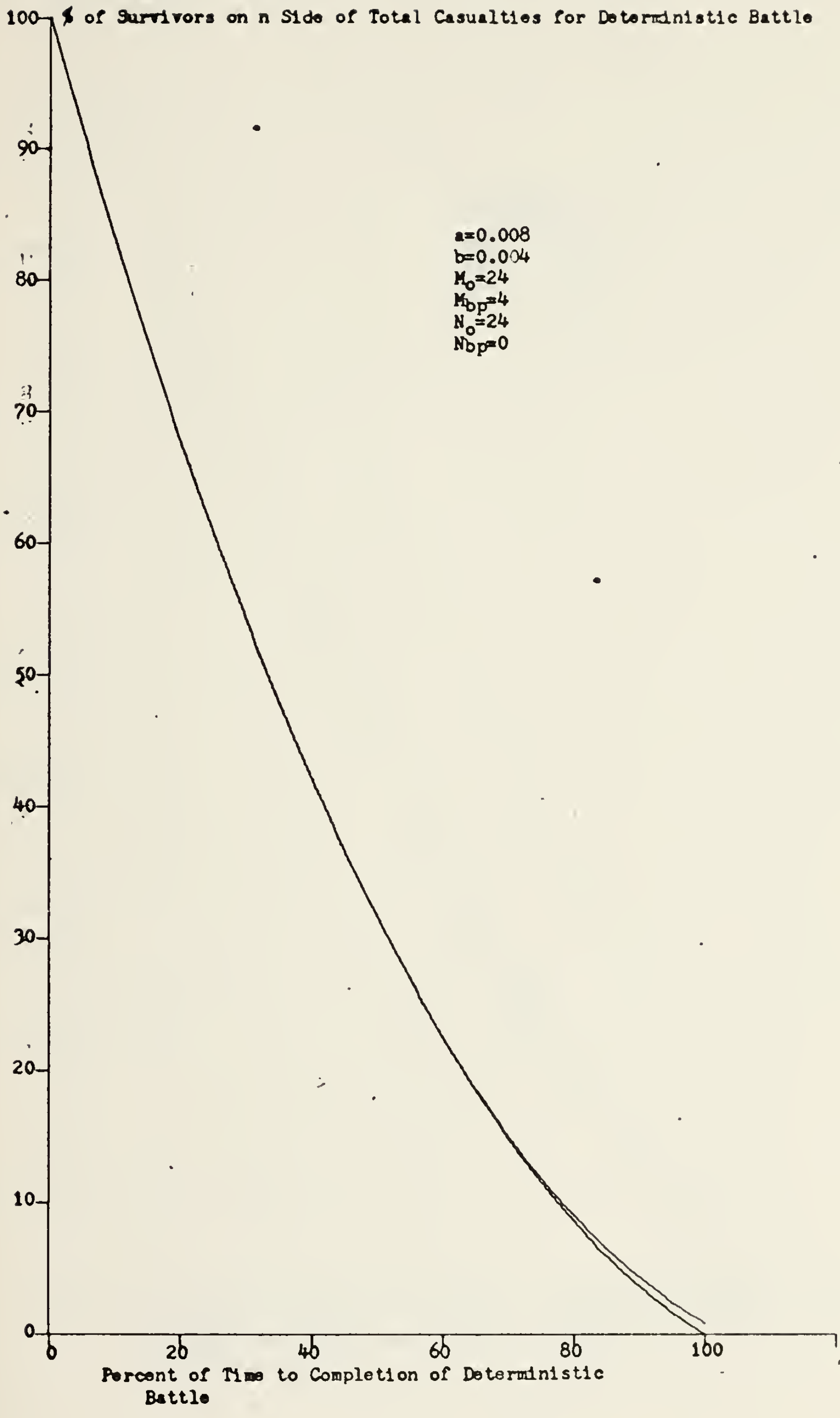
Time History Showing Bias

FIGURE 64



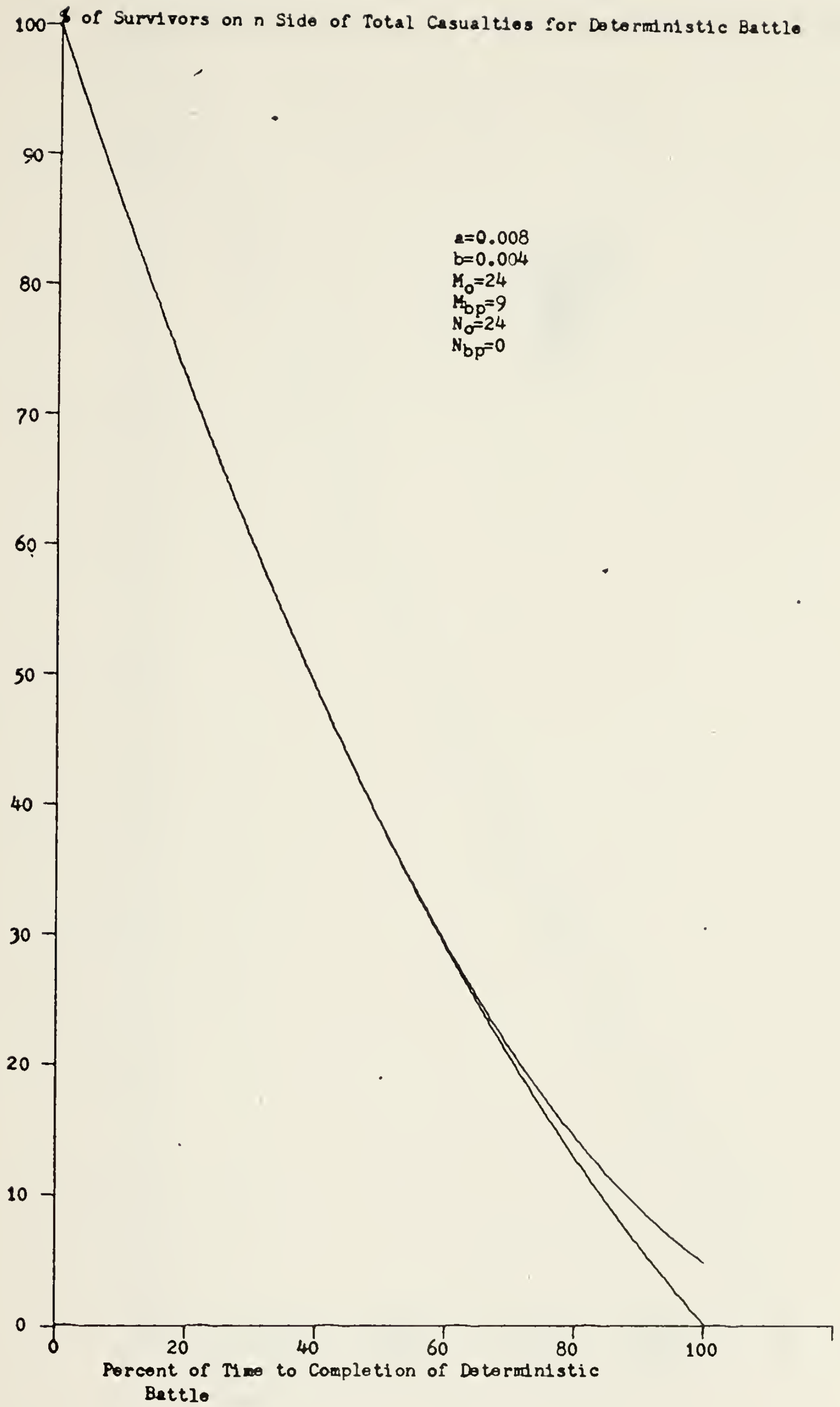
Time History Showing Bias

FIGURE 65



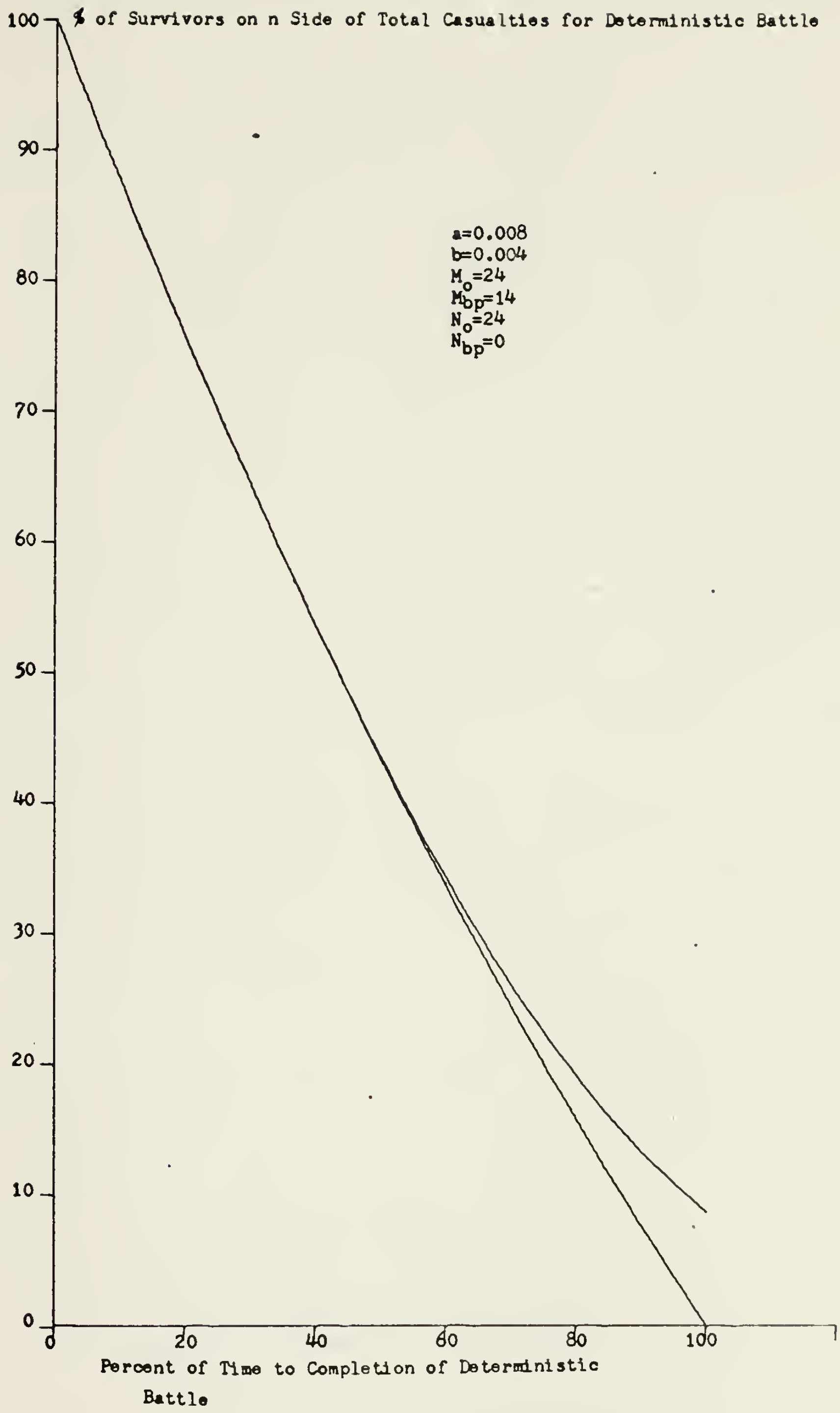
Time History Showing Bias

FIGURE 66



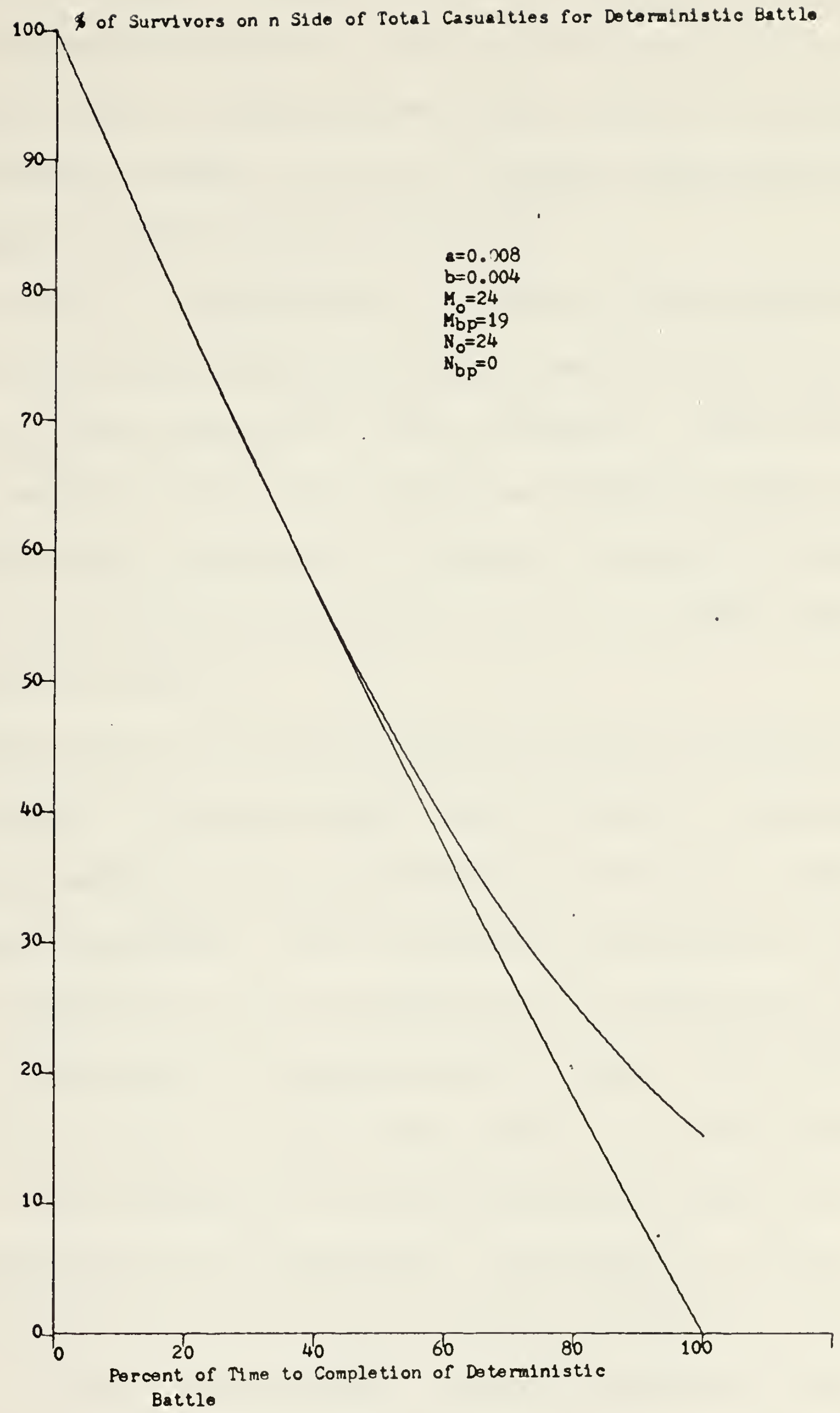
Time History Showing Bias

FIGURE 67



Time History Showing Bias

FIGURE 68



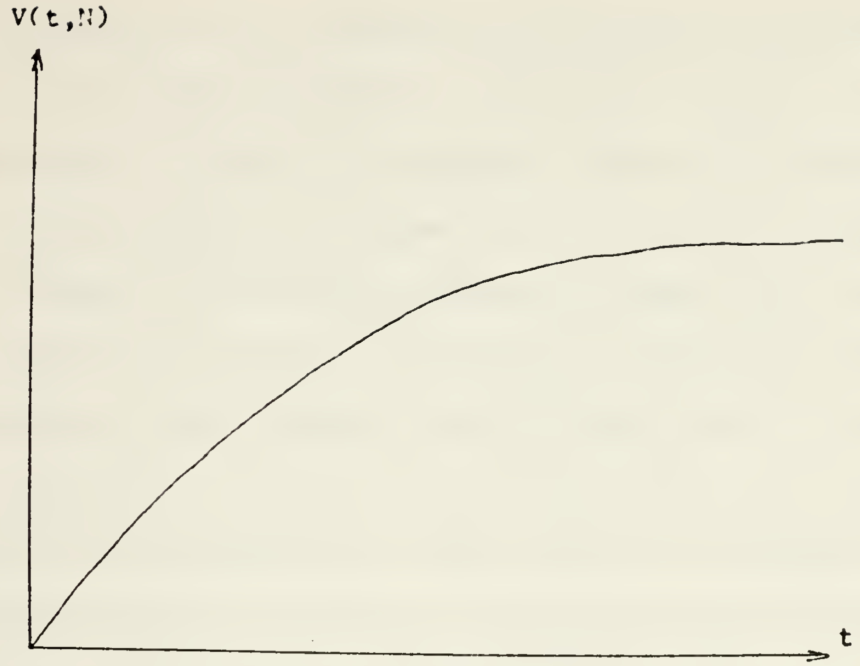
Time History Showing Bias

FIGURE 69

If the system being evaluated includes measures of effectiveness that are functions of the distribution of the final force levels, then in the case of large relative variance, the deterministic model is not an adequate representation of the complex random process of combat.

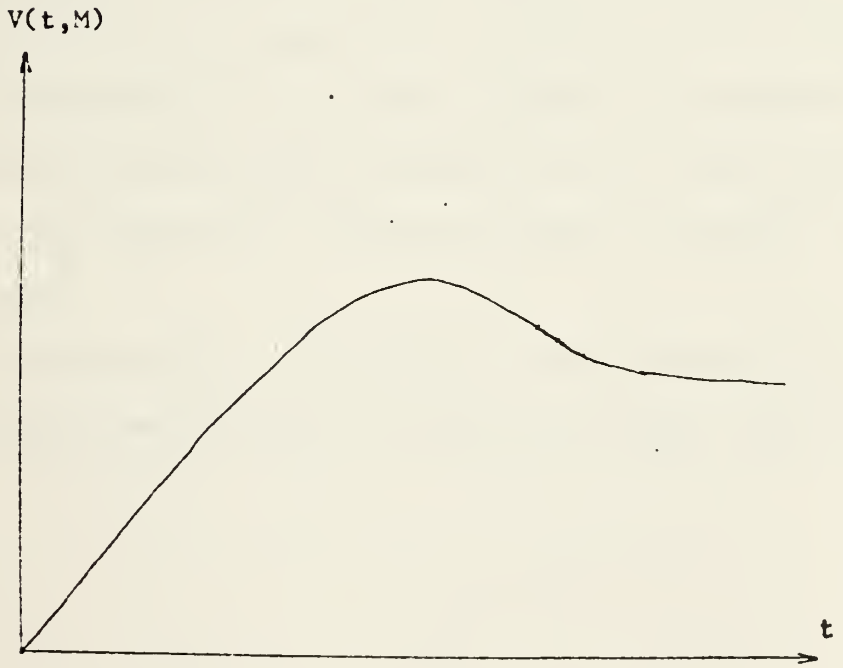
Clark [Ref. 8] analyzed the variance of the square-law stochastic attrition process and hypothesized there are two types of time histories of the variance. The first is where the variance grows and converges asymptotically to a final level, while the second is where it grows to a peak level and then decreases asymptotically to a final level (see Figures 70 and 71).

The qualitative pictures of the time state probabilities in Chapter II provide some useful insights into why the variance asymptotically approaches a value for every set of input parameters. As was noted, the change in the time state probability is a combination of diffusion and convective transport of the probability mass. The diffusion provides the variance in the force level. As the probability mass "hits" the boundary it remains there (is absorbed); for the probability mass absorbed, the diffusion effect has also ended and the variance of that probability mass is fixed. As more is absorbed, the less relative impact the remaining probability mass will have on the variance. Eventually, as time grows, the remaining probability will have no effect, thus giving the asymptotic characteristic.



VARIANCE

Figure 70

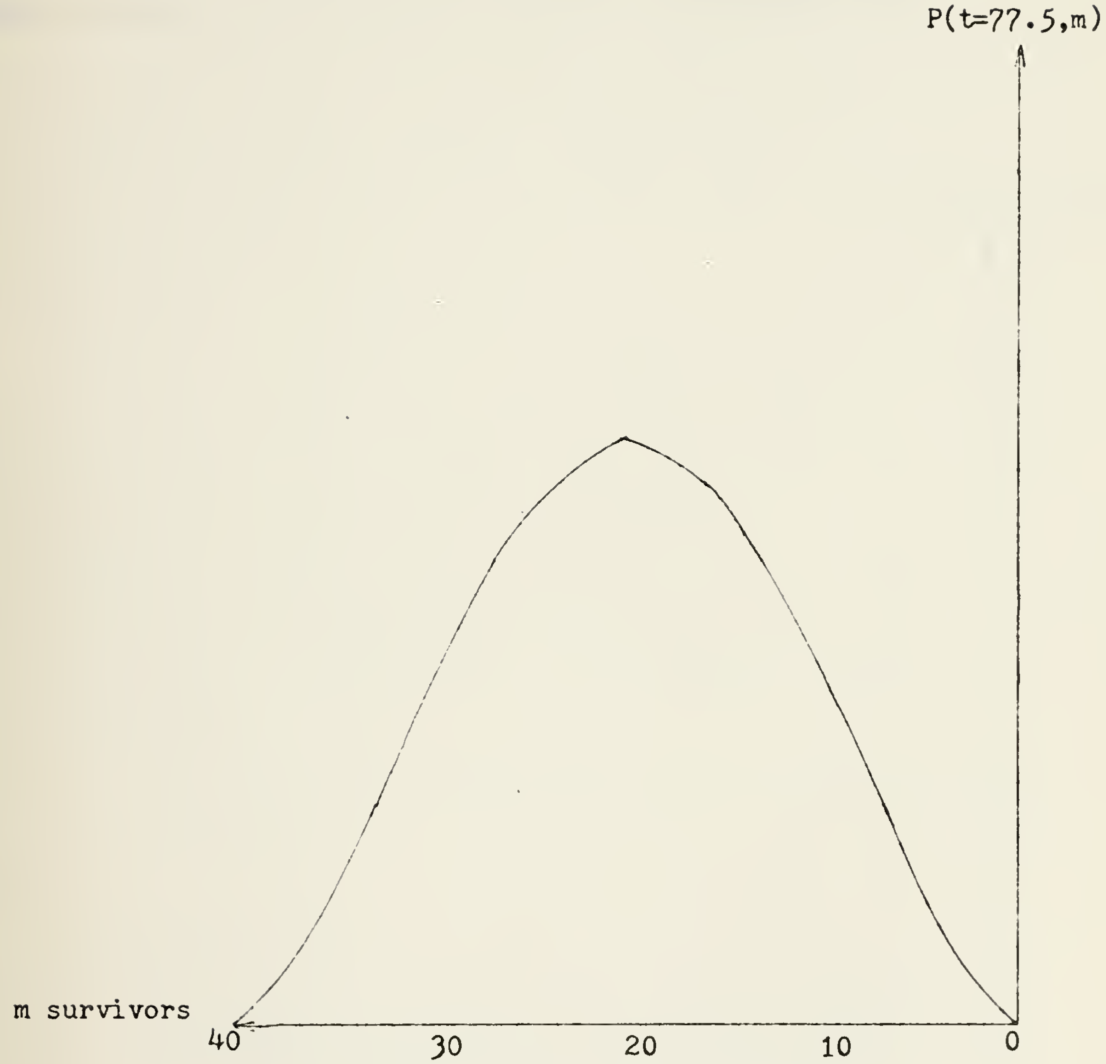


VARIANCE

Figure 71

However, because of the convective transport of probability mass, the remaining probability mass will not only continue to be diffused, it will, in general, arrive at the boundary closer to the breakpoint values (e.g., for the X force, the center of mass of the small probability mass, ΔP_1, arriving in the interval $(t, t+\Delta t_1)$, will be at m survivors, but ΔP_2 arriving in the interval $(t+\Delta t_1, t+\Delta t_2)$ will be centered at m-k survivors). Thus, as time increases, the convective transport of the probability mass causes the average of the probability mass not absorbed to approach the breakpoint force level. The breakpoint force level acts as a boundary, and the probability mass cannot be diffused past that point. In fact, the probability that would have been diffused past the breakpoint "piles up" at it, and the variance is not quite as great as it might be. In fact, if the probability mass is transported fast enough toward the breakpoint, the variance of the remaining probability mass may decrease in spite of the diffusion and might cause the overall variance to decrease.

For illustration, take Figure 3 and collapse the probability mass to the X (m survivor) axis. The result is:

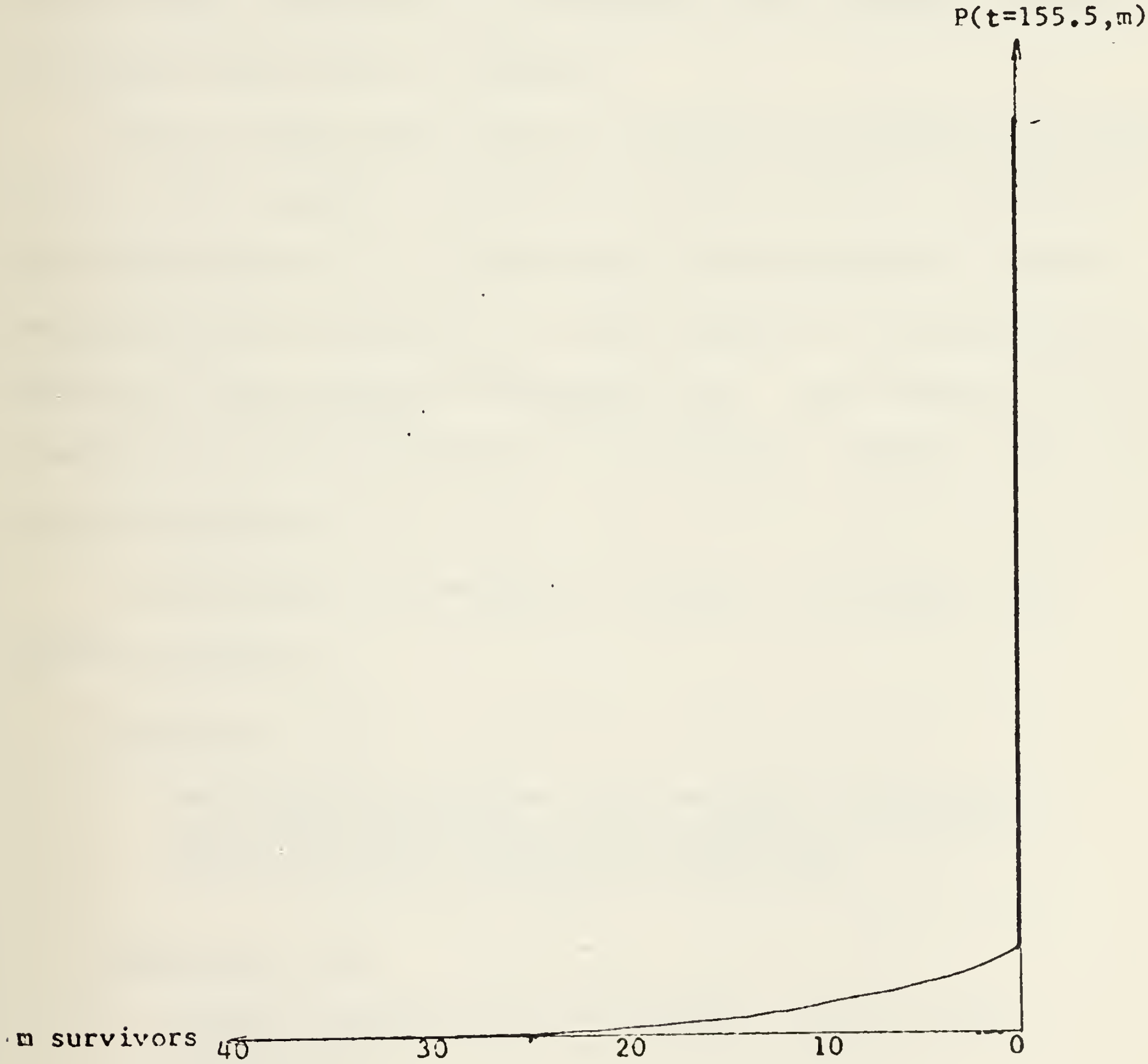


Collapsed Probability Mass

FIGURE 72

This gives a qualitative picutre of P(t,77.5,m). The variance is calculated from the sum of the squared deviations from the mean. In this case, it appears the variance will be "large" (in fact, it is on the order of magnitude of the mean).

If we now take Figure 5 and collapse the probability mass to the X (m survivor) axis, the result is:



COLLAPSED PROBABILITY MASS

Figure 73

Obviously the variance is much small than it was at $t = 77.5$.

Note that X is the decisive loser in this battle; the convective transport has moved the probability mass toward the breakpoint fast enough to offset the diffusion effect, and the variance has decreased. On the other hand, if the same graphs were made for the Y force (the decisive winner) it would be obvious that the probability mass is not being transported toward the Y breakpoint very rapidly, and the variance continues to increase.

Thus it seems that how the variance progresses is dependent on how rapidly the side is being attrited, relative to the other side. If a side wins, the variance will grow asymptotically, whereas if a side loses, the variance will peak and then decrease asymptotically. The results of all combinations of parameters in this study indicate this type of performance.

Furthermore, the results in Table V indicated two general trends.

Hypothesis 3-1

> As $m_o - m_{bp}$ increases, the variance increases in numerical value, but decreases in relative measure to the expected casualties.

Hypothesis 3-2

> As the forces approach parity, the variance increases.

a	m_o	m_{bp}	b	n_o	n_{bp}	$\sqrt{V(X)}$	$\sqrt{V(Y)}$
0.008	40	0	0.004	40	0	4.50	4.67
0.008	40	8	0.004	40	8	3.66	4.18
0.008	40	16	0.004	40	16	2.97	3.61
0.008	40	24	0.004	40	24	2.32	2.94
0.008	40	32	0.004	40	32	1.71	2.04
0.008	24	0	0.004	24	0	3.45	3.59
0.008	24	4	0.004	24	4	2.90	3.27
0.008	24	9	0.004	24	9	2.34	2.81
0.008	24	14	0.004	24	14	1.85	2.26
0.008	24	19	0.004	24	19	1.35	1.56
0.008	12	0	0.004	12	0	2.39	2.49
0.008	24	0	0.004	24	4	2.90	3.27
0.008	24	0	0.004	24	9	2.34	2.82
0.008	24	0	0.004	24	14	1.82	2.29
0.008	24	0	0.004	24	19	1.33	1.65
0.004	40	16	0.0015	40	16	2.89	2.94
0.004	40	24	0.0015	40	24	2.29	2.46
0.004	40	32	0.0015	40	32	1.68	1.81
0.004	24	0	0.0015	24	0	3.17	2.79
0.004	24	4	0.0015	24	4	2.73	2.60
0.004	24	9	0.0015	24	9	2.28	2.29
0.004	24	14	0.0015	24	14	1.77	1.92
0.004	24	19	0.0015	24	19	1.28	1.40
0.004	12	0	0.0015	12	0	2.20	1.96

TABLE V. Standard Deviation of Force Levels

The second trend is in general agreement with previous conclusions in the respect that, unless the forces are near parity, the models are not significantly different. The first trend is in agreement with Hypothesis 2-1, so that the conclusions reached for it seem to apply also for the variance. Thus, even though the variance is a different measure for comparison of the two models, it seems to agree with previous conclusions.

IV. SUMMARY

The intent of this analysis was to consider the question, can the complex random process of combat be adequately represented by a deterministic model. The models chosen for comparison were the deterministic and stochastic attrition versions of the Lanchester square-law attrition process. Three aspects of these processes have been compared and analyzed.

	Deterministic Model	Stochastic Model
A.	Fixed winner and loser	Probability of winning
B.	Time history of the force levels	Time history of the expected force levels
C.	No variance of the force levels	Variance of the force levels

Conclusions in the form of hypotheses were discussed for each of the above comparisons:

A. PROBABILITY OF WINNING

Hypothesis 1-1

> As the breakpoint force levels, m_{bp} and n_{bp} are moved closer to the initial force levels, m_o and n_o, respectively, the difference in the probability of winning between the models increases.

Hypothesis 1-2

> For fixed breakpoint casualty/initial force ratios (F_X and F_Y), the difference between the probability of winning for the deterministic model and the stochastic model decreases as the initial force levels increase.

Hypothesis 1-3

As the forces move away from parity, the difference between the probability of winning for the two models becomes negligible.

B. TIME HISTORY OF EXPECTED FORCE LEVELS

Hypothesis 2-1

Given fixed initial force levels and fixed attrition coefficients, as the breakpoint force levels increase, the numerical bias decreases. But as a percentage of casualties of the deterministic model, the bias increases.

Hypothesis 2-2

For fixed F_X and F_Y, and fixed attrition rate coefficients, the larger the initial forces, the greater the numerical bias, but the smaller the percentage bias.

Hypothesis 2-3

If any parameter is varied to bring the forces closer to parity, the time of the battle increases and the bias increases.

Hypothesis 2-4

If a battle were to be terminated at a time $t \leq 0.5\ t_f$, where t_f is the completion time of the deterministic model, there is no significant difference between the results of the two models.

C. VARIANCE OF FORCE LEVELS

Hypothesis 3-1

As $m_o - m_{bp}$ increases, the variance increases in numerical value, but decreases in relative measure to the expected casualties.

Hypothesis 3-2

> As the forces approach parity, the variance increases.

The hypotheses were formulated and evaluated through analyses of combinations of analytical solutions, numerical solutions, and graphical representations.

V. DISCUSSION

The hypotheses formulated in this thesis provide a basis for the extension of several conclusions about the adequacy of the deterministic model. When one side is going to win decisively in the deterministic model, the models do not differ significantly and the deterministic model is to be preferred. How decisively one side must win to make the difference insignificant is a function of two sets of parameters; the initial force levels and the breakpoint force levels. As the initial force levels increase, the relative differences between models decrease as long as m_o-m_{bp} and n_o-n_{bp} do not increase. Thus, with relatively large initial force levels (greater than 20) and breakpoint force levels such that there will be relatively large losses allowed (also greater than 20), the models are not significantly different.

It must be recognized that these conclusions only apply to the very simple and idealistic Lanchester square-law attrition process and the square-law stochastic attrition process. As is always the case, the insights into combat dynamics obtained from these models are no more valid than the models themselves. However, intuition indicates the conclusions may be generalized. It is intuitively appealing to say that, no matter what the model or degree of complexity, if the deterministic model predicts that one side will win

decisively, the stochastic model will not disagree significantly.

Nevertheless, it is felt that the hypotheses may provide a point of departure for the comparison of more complex models of combat.

VI. CONCLUSIONS

For the idealized model (i.e., the so-called Lanchester "square-law" attrition process), it is concluded that the complex random process of combat can be adequately represented by a deterministic model if the following conditions are met:

1. Each side starts with at least 20 combatants.
2. Each side is willing to take at least 20 casualties.
3. The forces are not near parity.
4. If the forces are near parity, but each side initially has in excess of 40 combatants and is willing to take in excess of 20 casualties.

Consequently, it seems plausible that similar results would be obtained if one were to compare analogous deterministic and stochastic "real world" combat models (i.e., large-scale complex system models of military forces). In other words, it seems reasonable to expect that if the opposing military forces are not near parity and have relatively "large" numbers of initial combatants, a deterministic model may be used to model the dynamics of combat without losing any essential information about the battle's outcome. From the research reported here, the author concludes that one may safely take "large" numbers of initial forces (in particular, of tanks), to mean units of battalion size or larger. Thus, it appears as though current U.S. Army deterministic models such as the Bonder/IUA model are to be

preferred for sensitivity analysis because of their computational advantages, once they are "calibrated" from Monte Carlo simulation output (for example, from DYNTACS or CARMONETTE).

APPENDIX A

COMPARISON OF TWO METHODS OF OBTAINING THE PROBABILITY OF WINNING

In this appendix it is shown that two apparently different methods of obtaining the win probabilities are really the same. In his doctoral thesis, Springall [Ref. 28] derived a recursive expression for the probability of winning in a homogeneous force, constant coefficient stochastic Lanchester model. His expressions and the development of them are quite different from those of Brown [Ref. 7]. Brown developed the familiar set of partial difference equations:

$$P(m,n:r,s) = \frac{A(r,s)}{A(r,s) + B(r,s)} P(m,n:r-1,s) + \frac{B(r,s)}{A(r,s) + B(r,s)} P(m,n:r,s-1) \quad (66)$$

with boundary conditions

$$P(m_{bp},n:m_{bp},s) = \begin{cases} 1 & n = s \\ 0 & \text{otherwise} \end{cases} \quad (67)$$

$$P(m,n_{bp}:r,n_{bp}) = \begin{cases} 1 & m = r \\ 0 & \text{otherwise} \end{cases} \quad (68)$$

$$P(m_{bp},n:r,s) = 0 \quad n > s; \qquad P(m,n_{bp}:r,s) = 0 \quad m > r \quad (69)$$

$$P(m_{bp},n_{bp}:r,s) = \begin{cases} 1 & m_{bp} = r \text{ and } n_{bp} = s \\ 0 & \text{otherwise} \end{cases} \tag{70}$$

where $P(m,n_{bp}:m_o,n_o)$ = probability X will win with m survivors. When this is summed from $m_{bp}+1$ to m_o, it yields the probability X will win:

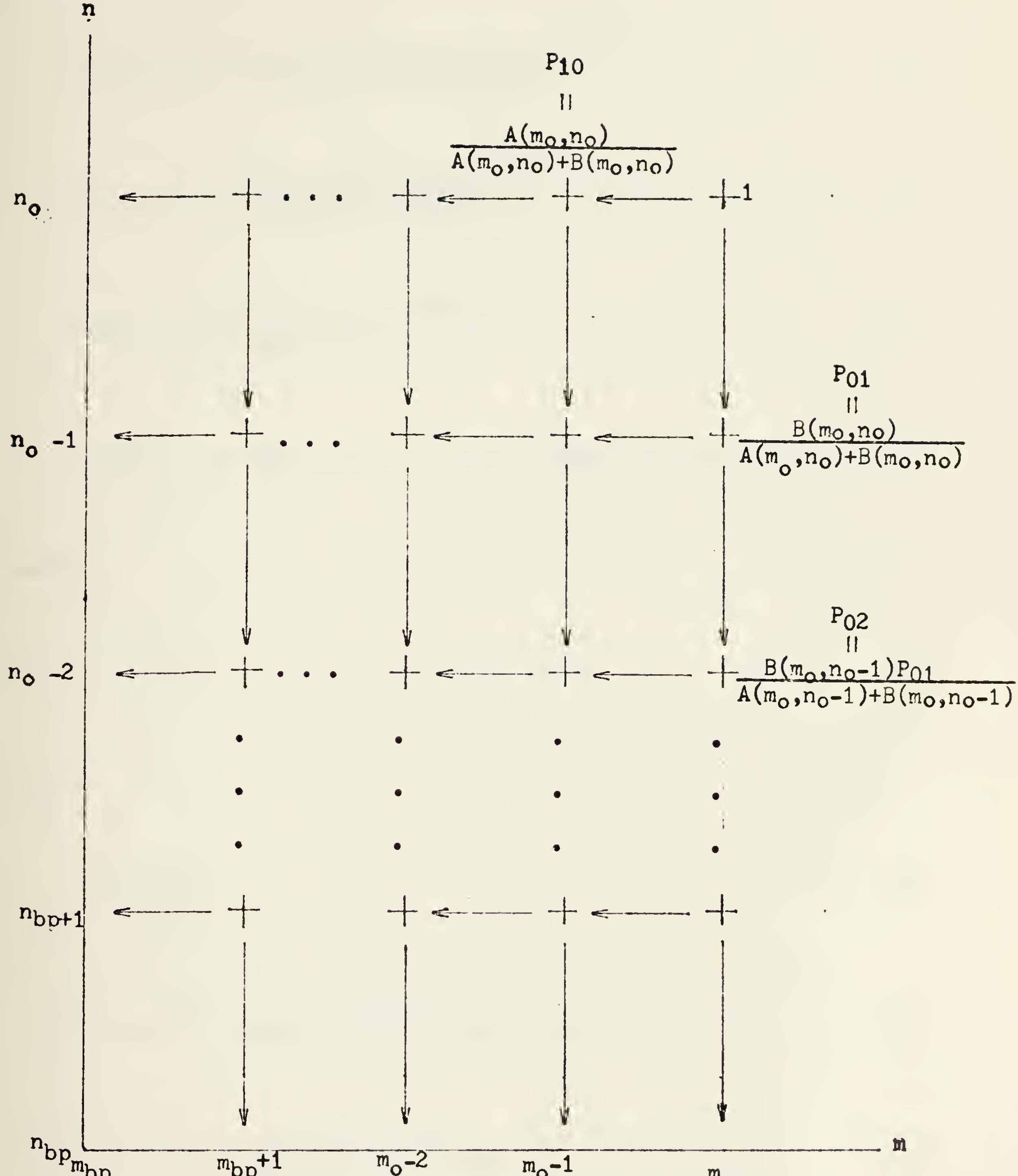
$$P_X = \sum_{m=m_{bp}+1}^{m_o} P(m,n_{bp}:m_o,n_o) \tag{71}$$

When the above equations are looked at carefully, what they are doing is explained by the following figure (Fig. 74).

The probability that X will win with m survivors is simply the sum of the probabilities of all possible paths from (m_o,n_o) to (m,n_{bp}) with the probability of going from state (r,s) to state (r-1,s) equal to $\frac{A(r,s)}{A(r,s) + B(r,s)}$ and the probability of going from state (r,s) to state (r,s-1) equal to $\frac{B(r,s)}{A(r,s) + B(r,s)}$. And the probability X will win is simply the probability that X will win with m_o survivors, or m_o-1 survivors, or ..., or $m_{bp}+1$ survivors.

A similar analysis of Springall's equations will show that the two methods are identical. Springall concludes that

$$P(m,n_{bp}:m_o,n_o) = B(m,n_{bp}+1)X(m,n_{bp}+1) \tag{72}$$



Method of Solving Recursive Relationship for the Probability of Winning for Brown's Equations

FIGURE 74

where $X(m,n_{bp}+1)$ is derived from the following set of recursive relations;

$$X(m_o,n_o) = \frac{1}{A(m_o,n_o) + B(m_o,n_o)} \tag{73}$$

$$X(m_o,n) = \frac{B(m_o,n+1)X(m_o,n+1)}{A(m_o,n) + B(m_o,n)} \qquad n_{bp} < n < n_o \tag{74}$$

$$X(m,n_o) = \frac{A(m+1,n_o)X(m+1,n_o)}{A(m,n_o) + B(m,n_o)} \qquad m_{np} < m < m_o \tag{75}$$

$$X(m,n) = \frac{A(m+1,n)X(m+1,n) + B(m,n+1)X(m,n+1)}{A(m,n) + B(m,n)} \tag{76}$$

Define

$$P'(r,s) = (A(r,s) + B(r,s))X(r,s) \tag{77}$$

Then

$$P'(m_o,n_o) = 1 \tag{78}$$

$$P'(m_o,n) = \frac{B(m_o,n+1)}{A(m_o,n+1) + B(m_o,n+1)} P'(m_o,n+1) \tag{79}$$

$$P'(m,n_o) = \frac{A(m+1,n_o)}{A(m+1,n_o) + B(m+1,n_o)} P'(m+1,n_o) \tag{80}$$

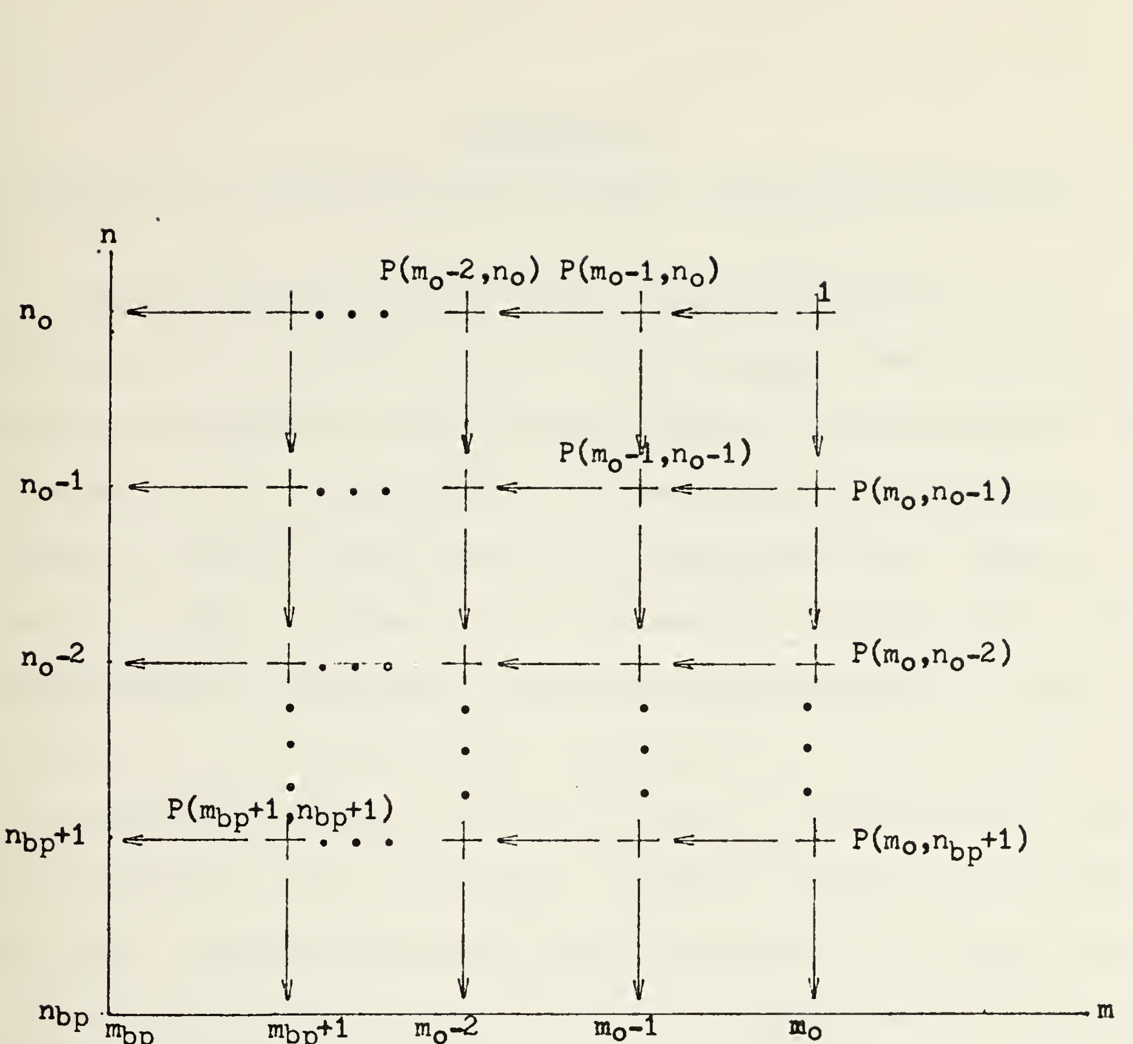
$$P'(m,n) = \frac{A(m+1,n)P'(m+1,n)}{A(m+1,n) + B(m+1,n)} + \frac{B(m,n+1)P'(m,n+1)}{A(m,n+1) + B(m,n+1)} \tag{81}$$

Therefore

$$P(m_{bp},n:m_o,n_o) = \frac{A(m_{bp}+1,n)P'(m_{bp}+1,n)}{A(m_{bp}+1,n) + B(m_{bp}+1,n)} \tag{82}$$

$$P(m,n_{bp}:m_o,n_o) = \frac{A(m,n_{bp}+1)P'(m,n_{bp}+1)}{A(m,n_{bp}+1) + B(m,n_{bp}+1)} \tag{83}$$

What these equations say is, the probability of starting with (m_o,n_o) survivors is one. The probability of having one more casualty from X is $\frac{A(m_o,n_o)}{A(m_o,n_o) + B(m_o,n_o)}$ and from Y is $\frac{B(m_o,n_o)}{A(m_o,n_o) + B(m_o,n_o)}$. The probability of having (m,n) survivors is the probability of having (m+1,n) survivors times the probability of an additional casualty being from X, added to the probability of having (m,n+1) survivors times the probability of an additional casualty being from Y. This recursive relationship leads to the following figure (Figure 75), which is the same as Figure 74. Thus, even though the sets of relationships appear to differ, they do yield the same results. This was further supported by a comparison of results from Lee and Wannasilpa [Ref. 20] and the results from the enclosed computer program, which was set up using the recursive equations of Springall. For the same parameters, the results were the same within 0.00001.



Method of Solving Recursive Relationship for the Probability of Winning for Springall's Equations

FIGURE 75

The differences are assumed to have occurred because of roundoff errors within the computer system used.

The program accuracy was further checked by

$$\sum_{m=m_{bp}+1}^{m_o} P(m,n_{bp}:m_o,n_o) + \sum_{n=n_{bp}+1}^{n_o} P(m_{bp},n:m_o,n_o) = 1 \qquad (84)$$

All results were in the interval (0.9999692,1.0000000).

APPENDIX B

NUMERICAL SOLUTION OF THE FORWARD KOLMOGOROV EQUATIONS

Exact solutions for the time state probabilities for the square-law stochastic attrition process can be developed only for specific initial force levels. To the author's best knowledge, no solution for the time state probabilities for general initial force levels has been developed. Even if one were able to develop such a general solution, the result would be of no practical use due to its complexity. The solution for the linear-law stochastic attrition process was developed by Clark [Ref. 8], but it is too complex for practical use: So, solutions, if they could be developed, for more complex Lanchester-type stochastic attrition processes, would most likely be of no practical use. Therefore, the most practical way to solve such equations is through numerical integration techniques. The method chosen for this thesis was the fourth order Runga-Kutta method [Ref. 21]. The computer program is set up to solve Lanchester-type attrition process models of the form

$$\frac{dP(t,m,n)}{dt} = A(m+1,n)P(t,m+1,n) + B(m,n+1)P(t,m,n+1) - (A(m,n) + B(m,n))P(t,m,n) \qquad (85)$$

with

$$P(0,m,n) = \begin{cases} 1 & n = n_o \quad \text{and} \quad m = m_o \\ 0 & \text{otherwise} \end{cases} \tag{86}$$

$$P(t,m,n) = 0 \quad m < m_o \quad \text{or} \quad n < n_o \tag{87}$$

and boundary conditions

$$\frac{dP(t,m,n_{bp})}{dt} = B(m,n_{bp}+1)P(t,m,n_{bp}+1) \tag{88}$$

$$\frac{dP(t,m_{np},n)}{dt} = A(m_{bp}+1,n)P(t,m_{bp}+1,n) \tag{89}$$

However, with a few minor alterations, the program will solve the more general Lanchester-type stochastic attrition processes with attrition rates of the form $F(t,m,n)$ and $G(t,m,n)$.

Numerical precision of the Runga-Kutta methods is dependent on the length of the time step chosen. For the time steps chosen, numerical precision appeared to be satisfactory since

$$\sum_{m=m_{bp}}^{m_o} \sum_{n=n_{bp}}^{n_o} P(t,m,n) \text{ was always in the interval}$$

(0.9992,1.0003).

Because the expected number of survivors was calculated from

$$\bar{m}(t) = \sum_{m=m_{bp}}^{m_o} \sum_{n=n_{bp}}^{n_o} mP(t,m,n) \tag{90}$$

it was felt the solutions were accurate enough. Additionally, because exact solutions exist for general initial force levels for $P(t,m_o,n)$ and $P(t,m,n_o)$, they were used to check the accuracy of the approximate solutions. The approximate solutions never deviated from the exact solution by more than 0.00005.

APPENDIX C

SCALING

In a comparison of a stochastic model and the equivalent deterministic model, two classes of comparisons need to be considered. The first is the difference in results obtained from the two models. The second is to take these differences and see how they change with variations in parameters. One runs into difficulties when trying to make sense out of these comparisons when results are expressed in dimensional units (i.e., What is the significance of the fact that at 150 minutes into a battle, the difference in survivors is 1.8 tanks?).

One method of overcoming this problem is the technique of scaling. If comparisons are made when results are in dimensionless quantities, it is much easier to see if there are any significant differences. For example, one can easily compare two probability statements and note any difference between them.

Scaling is a subset of a "science" called dimensional analysis, a technique often used in the physical sciences. For example, in the area of fluid dynamics, Reynolds deduced from some experiments that laminar flow broke down into turbulent flow at some critical velocity above that at which turbulent flow was restored to the laminar condition;

the former velocity being called the upper critical velocity, and the latter, the lower critical velocity. Reynolds was able to generalize his conclusions by the introduction of a dimensionless term called the Reynold's number. He found that certain critical values of the Reynold's number defined the upper and lower critical velocities for <u>all</u> fluids flowing in <u>all</u> sizes of pipes, and thus deduced that single numbers define the limits of laminar and turbulent pipe flow for all fluids. Many other dimensionless numbers are used to generalize results — the Froude number, the Cauchy number, the Mach number, etc. Even in mathematics dimensionless quantities are used to generalize results — π and e, for example. (π is the ratio between the circumference of a circle and its diameter, regardless of the units of measure employed.)

Scaling involves the ratio of two quantities with the same dimensions. The dimension of a ratio of two quantities with the same dimensions is said to be dimensionless. The analysis in this paper is concerned with two classes of dimensions; time and quantity (number of casualties/survivors, and the differences in casualties/survivors). Either model (the stochastic or deterministic) could have been chosen as the basis for forming ratios for the two classes of dimensions; the deterministic model was used as a basis. Time was transformed into a dimensionless quantity by using the ratio t/t_f, where t_f is the time at which the deterministic battle would

end. It is easy to see that if $0 \leq t \leq t_f$, then the ratio is between zero and one.

The transformation of the quantities requires some explanation of the rationale behind it. The transformation used was

$$\Delta_{ratio} = 1 - \frac{\text{number of casualties at } t}{\text{number of casualties at } t_f \text{ for the deterministic mode}}$$

It is easily seen that, for the deterministic model with $0 \leq t \leq t_f$; $0 \leq \Delta_{ratio} \leq 1$. A dimensionless quantity was desired that would express the "importance" of the number of casualties. A difference of one casualty out of forty is not as "important" as a difference of one casualty out of four. Δ_{ratio} expresses this "importance". For example, at t_f, $\Delta_{ratio} = 0$ for the deterministic model, while $\Delta_{ratio} \neq 0$, in general, for the stochastic model. In fact,

$$\Delta_{ratio} = \frac{\text{the difference in casualties at } t_f}{\text{number of casualties at } t_f \text{ for the deterministic model}}.$$

as the number of casualties at t_f for the deterministic model increases, Δ_{ratio} decreases, giving the desired result.

APPENDIX D

DERIVATION OF EXPRESSION FOR BIAS IN THE SQUARE LAW STOCHASTIC ATTRITION PROCESS WITH FIXED BREAKPOINT FORCE LEVELS

From previous results it is known that the square-law stochastic attrition process is defined by:

for $m_{bp}+1 \leq m \leq m_o$ and $n_{bp}+1 \leq n \leq n_o$;

$$\frac{dP(t,m,n)}{dt} = anP(t,m+1,n) + bmP(t,m,n+1) - (an+bm)P(t,m,n) \tag{91}$$

and on the boundaries of the state space,

$$\frac{dP(t,m,n_{bp})}{dt} = bmP(t,m,n_{bp}+1) \tag{92}$$

$$\frac{dP(t,m_{np},n)}{dt} = anP(t,m_{bp}+1,n) \tag{93}$$

also,

$$P(t,m,n) = 0 \quad \text{for } m > m_o \quad \text{and/or} \quad n > n_o$$

and

$$P(0,m,n) = \begin{cases} 1 & m = m_o \text{ and } n = n_o \\ 0 & \text{otherwise} \end{cases}$$

Define

$$\Delta_m P(t,m,n) = P(t,m+1,n) - P(t,m,n)$$

and

$$\Delta_n P(t,m,n) = P(t,m,n+1) - P(t,m,n)$$

so that (91) may be written as

for $m_{bp}+1 \le m \le m_o$ and $n_{bp}+1 \le n \le n_o$

$$\frac{dP(t,m,n)}{dt} = an\Delta_m P(t,m,n) = bm\Delta_n P(t,m,n) \qquad (94)$$

First consider two jointly distributed discrete random variables, M and N with probability mass denoted as $f_{M,N}(m,n)$. Further, assume that $f_{M,N}(m,n) = 0$ for $m < m_{bp}$ and $n < n_{bp}$. Then the expected value of M, denoted as E(M) is given by

$$E(M) = \sum_{m=m_{bp}}^{\infty} \sum_{n=n_{bp}}^{\infty} m f_{M,N}(m,n)$$

This may be expressed in terms of the conditional expectation

$$E(M:N=n) = \frac{\sum_{m=m_{bp}}^{\infty} mf_{M,N}(m,n)}{f_N(n)}$$

as $E(M) = \sum_{n=n_{bp}}^{\infty} f_N(n)E(M:N=n)$

Note that

$$\frac{d\overline{m}}{dt} = [\frac{d}{dt} \sum_{m=m_{bp}}^{m_o} \sum_{n=n_{bp}}^{n_o} mP(t,m,n)] \tag{95}$$

To compute $\frac{d\overline{m}}{dt}$, which is given by equation (101) below, first sum equation (94) over all n, from $n_{bp}+1$ to n_o to obtain

$$\frac{d}{dt} [\sum_{n=n_{bp}+1}^{n_o} P(t,m,n)] = a \sum_{n=n_{bp}+1}^{n_o} n\Delta_m P(t,m,n)$$

$$+ bm \sum_{n=n_{bp}+1}^{n_o} \Delta_n P(t,m,n) \tag{96}$$

Note that

$$\sum_{n=n_{bp}+1}^{n_o} \Delta_n P(t,m,n) = \sum_{n=n_{bp}+1}^{n_o} [P(t,m,n+1) - P(t,m,n)]$$

$$= P(t,m,n_o+1) - P(t,m,n_{bp}+1)$$

or

$$\sum_{n=n_{bp}+1}^{n_o} \Delta_n P(t,m,n) = - P(t,m,n_{bp}+1)$$

Thus

$$\frac{d}{dt}\left[\sum_{n=n_{bp}+1}^{n_o} P(t,m,n)\right] = a\left[\sum_{n=n_{bp}+1}^{n_o} n\Delta_m P(t,m,n)\right]$$

$$- bmP(t,m,n_{bp}+1) \qquad (97)$$

Adding (92) to (97) results in

$$\frac{d}{dt}\left[\sum_{n=n_{bp}}^{n_o} P(t,m,n)\right] = a\left[\sum_{n=n_{bp}+1}^{n_o} n\Delta_m P(t,m,n)\right] \qquad (98)$$

Multiply (98) by m and sum over m from $m_{bp}+1$ to m_o, and add (93) multiplied by m_{bp}:

$$\frac{d}{dt}\left[\sum_{m=m_{bp}}^{m_o} \sum_{n=n_{bp}}^{n_o} mP(t,m,n)\right] = \frac{d\bar{m}}{dt}$$

$$= a\left[\sum_{m=m_{bp}+1}^{m_o} \sum_{n=n_{bp}+1}^{n_o} mn\Delta_m P(t,m,n)\right] + m_{bp} \sum_{n=n_{bp}}^{n_o} anP(t,m_{bp}+1,n)$$

$$(99)$$

Observe that

$$\sum_{m=m_{bp}+1}^{m_o} \sum_{n=n_{bp}+1}^{n_o} mn\Delta_m P(t,m,n)$$

$$= \sum_{m=m_{bp}+1}^{m_o} m\Delta_m \sum_{n=n_{bp}+1}^{n_o} nP(t,m,n)$$

$$= \sum_{m=m_{bp}+1}^{m_o} m[\sum_{n=n_{bp}+1}^{n_o} nP(t,m+1,n) - \sum_{n=n_{bp}+1}^{n_o} nP(t,m,n)]$$

$$= \sum_{m=m_{bp}+1}^{m_o} m \sum_{n=n_{bp}+1}^{n_o} nP(t,m+1,n) - \sum_{m=m_{bp}+1}^{m_o} m \sum_{n=n_{bp}+1}^{n_o} nP(t,m,n)$$

$$= \sum_{m=m_{bp}+2}^{m_o+1} (m'-1) \sum_{n=n_{bp}+1}^{n_o} nP(t,m',n)$$

$$- \sum_{m=m_{bp}+1}^{m_o} m \sum_{n=n_{bp}+1}^{n_o} nP(t,m,n)$$

$$= \sum_{m=m_{bp}+1}^{m_o+1} (m'-1) \sum_{n=n_{bp}+1}^{n_o} nP(t,m',n)$$

$$- \sum_{m=m_{bp}+1}^{m_o} m \sum_{n=n_{bp}+1}^{n_o} nP(t,m,n)$$

$$- m_{bp} \sum_{n=n_{bp}+1}^{n_o} nP(t,m_{np}+1,n)$$

$$= \sum_{m=m_{bp}+1}^{m_o} (m'-1) \sum_{n=n_{bp}+1}^{n_o} nP(t,m',n)$$

$$- \sum_{m=m_{bp}+1}^{m_o} m \sum_{n=n_{bp}+1}^{n_o} nP(t,m,n)$$

$$- m_{bp} \sum_{n=n_{bp}+1}^{n_o} nP(t,m_{bp}+1,n)$$

$$= \sum_{m=m_{bp}+1}^{m_o} (m'-1-m) \sum_{n=n_{bp}+1}^{n_o} nP(t,m,n)$$

$$- m_{np} \sum_{n_{bp}+1}^{n_o} nP(t,m_{np}+1,n)$$

$$= \sum_{m=m_{bp}+1}^{m_o} \sum_{n=n_{bp}+1}^{n_o} nP(t,m,n)$$

$$- m_{bp} \sum_{n=n_{bp}+1}^{n_o} nP(t,m_{bp}+1,n) \qquad (100)$$

Substituting (100) into (99)

$$\frac{d\overline{m}}{dt} = -a \sum_{m=m_{bp}+1}^{m_o} \sum_{n=n_{bp}+1}^{n_o} nP(t,m,n)$$

$$- am_{bp} \sum_{n=n_{bp}+1}^{n_o} nP(t,m_{bp}+1,n)$$

$$+ am_{bp} \sum_{n=n_{bp}}^{n_o} nP(t,m_{np}+1,n)$$

$$= -a \sum_{m=m_{bp}+1}^{m_o} \sum_{n=n_{bp}+1}^{n_o} nP(t,m,n) + am_{bp}n_{bp}P(t,m_{bp}+1,n_{bp})$$

$$= -a \sum_{m=m_{bp}+1}^{m_o} \sum_{n=n_{bp}}^{n_o} nP(t,m,n) + a \sum_{m=m_{bp}+1}^{m_o} n_{bp}P(t,m,n_{bp})$$

$$+ am_{bp}n_{bp}P(t,m_{bp}+1,n_{bp})$$

$$= -a \sum_{m=m_{bp}}^{m_o} \sum_{n=n_{bp}}^{n_o} nP(t,m,n) + a \sum_{n=n_{bp}}^{n_o} nP(t,m_{bp},n)$$

$$+ a \sum_{m=m_{bp}+1}^{m_o} n_{bp}P(t,m,n_{bp})$$

$$+ am_{bp}n_{bp}P(t,m_{bp}+1,n_{bp})$$

$$\frac{d\bar{m}}{dt} = -a\bar{n} + a[\sum_{n=n_{bp}}^{n_o} nP(t,m_{bp},n) + n_{bp}m_{bp}P(t,m_{bp}+1,n_{bp})$$

$$+ n_{bp} \sum_{m=m_{bp}+1}^{m_o} P(t,m,n_{bp})] \qquad (101)$$

Equivalently,

$$\frac{d\bar{n}}{dt} = -b\bar{m} + b[\sum_{m=m_{bp}}^{m_o} mP(t,m,n_{bp}) + m_{bp}n_{bp}P(t,m_{bp},n_{bp}+1)$$

$$+ m_{bp} \sum_{n=n_{bp}+1}^{n_o} P(t,m_{bp},n)] \tag{102}$$

```
C                           COMPUTER PROGRAM
C
C
C
C
C           SOLUTION TO STOCHASTIC LANCHESTER EQUATIONS
C           USING FOURTH ORDER RUNGA-KUTTA APPROXIMATIONS
C
C
C     LO=NUMBER OF TIME INTERVALS NEEDED FOR COMPLETE
C        SOLUTION TO DETERMINISTIC EQUIVALENT EQUATION
C     MO=NUMBER OF COMBATANTS X STARTS WITH
C     NO=NUMBER OF COMBATANTS Y STARTS WITH
C     H=SIZE OF TIME INCREMENT
C     A(X,Y)=''ATTRITION'' RATE FOR X IN TERMS OF X-1 & Y-1
C     B(X,Y)=''ATTRITION'' RATE FOR Y IN TERMS OF X-1 & Y-1
C     FX=BREAKPOINT FORCE RATIO FOR X
C     FY=BREAKPOINT FORCE RATIO FOR Y
C     MBP=(1-FX)*MO
C     NBP=(1-FY)*NO
C     DIMENSION KX(NKXY), KY(NKXY)
C     DIMENSION PROB(MO+1-MBP,NO+1-NBP)
C     DIMENSION WK(MO+1-MBP,NO+1-NBP,3)
C     DIMENSION D(LO+1),DETERM(LO+1)
C     DIMENSION P1(LO+1,MO+1),P2(LO+1,MO+1)
C     DIMENSION EST(LO+1), VAR(LO+1)
C     IF A PLOT OF THE TIME HISTORY OF THE X SIDE IS DESIRED
C        SET IT=0.  OTHERWISE SET IT=THE POINT IN TIME AT
C        WHICH A THREE DIMENSIONAL PLOT OF THE JOINT
C        PROBABILITY DISTRIBUTION OF X AND Y IS DESIRED
C        (NOTE- TIME=IT*H).
C
C
      DIMENSION P1(313,41),P2(313,41),EST(313),D(313),
     1F(2),SIZE(2),KX(100),KY(100),
     2PROB(41,41),WK(41,41,3),
     3VAR(313),DETERM(313)
      INTEGER X,Y,U,Z
      REAL*8 TTL(12)/'CRAIG J.',5*'        ','SMC 1462',5*'
      REAL XTITLE/'TIME'/,YTITLE/'SURV'/,ZTITLE/'    '/
      REAL K1,K2,K3,K4,K1A,K2A,K3A,K4A,K2B,K3B,K4B,K1C,K2C,
     1K1B,K1D,K2D,K3D,K4D,K1E,K2E,K3E,K4E,K1G,K2G,K3G,K4G,
     2K4F,K3C,K4C,K1F,K2F,K3F
C
C     LOGICAL*1 IDN(MO+1-MBP,NO+1-NBP)
C
      LOGICAL*1 IDN(41,41)
      F(1)=0.0
      F(2)=0.0
      SIZE(1)=7.0
      SIZE(2)=9.0
      LO=312
      MO=40
      NO=40
      NKXY=100
      MO1=MO+1
      LO1=LO+1
      NO1=NO+1
      IT=10
      A1=0.008
      B1=0.004
      H=2.0
      FX=1.0
      FY=1.0
      MBP=(1-FX)*MO
      NBP=(1-FY)*NO
      MS=MO+1-MBP
      NS=NO+1-NBP
      MBP1=MBP+1
      NBP1=NBP+1
      MS1=MS-1
  600 DO 13 L=1,LO1
      EST(L)=0.0
```

```
      VAR(L)=0.0
   13 CONTINUE
      DC 400 N=1,NS
      Y=NO+2-N
      DO 300 M=1,MS
      X=MO+2-M
      DO 210 L=1,LO1
      IF(L.EQ.1)GO TO 500
      IF(X.EQ.MO+1)GO TO 502
      IF(X.EQ.MBP+1)GO TO 504
      IF(Y.EQ.NO+1)GO TO 506
      IF(Y.EQ.NBP+1)GO TO 507
      K1=A(X+1,Y)*P1(L-1,X+1)+B(X,Y+1)*P2(L-1,X)-(A(X,Y)
     1+B(X,Y))*P1(L-1,X)
      K2=A(X+1,Y)*0.5*(P1(L,X+1)+P1(L-1,X+1))+B(X,Y+1)*0.5*
     1(P2(L,X)+P2(L-1,X))-(A(X,Y)+B(X,Y))*(P1(L-1,X)+H*K1
     2*0.5)
      K3=A(X+1,Y)*0.5*(P1(L,X+1)+P1(L-1,X+1))+B(X,Y+1)*0.5*
     1(P2(L,X)+P2(L-1,X))-(A(X,Y)+B(X,Y))*(P1(L-1,X)+H*K2
     2*0.5)
      K4=A(X+1,Y)*P1(L,X+1)+B(X,Y+1)*P2(L,X)-(A(X,Y)+B(X,Y))
     1*(P1(L-1,X)+H*K3)
      P1(L,X)=P1(L-1,X)+(H/6.0)*(K1+2.0*K2+2.0*K3+K4)
      GC TO 200
  500 IF(X.EQ.MO+1.AND.Y.EQ.NO+1)GO TO 501
      P1(L,X)=0.0
      GC TO 200
  501 P1(L,X)=1.0
      GO TO 200
  502 IF(Y.NE.NO+1)GO TO 503
      P1(L,X)=EXP(-(A(X,Y)+B(X,Y))*(L-1)*H)
      GC TO 200
  503 IF(Y.EQ.NBP+1)GO TO 509
      K1A=B(X,Y+1)*P2(L-1,X)-(A(X,Y)+B(X,Y))*P1(L-1,X)
      K2A=B(X,Y+1)*0.5*(P2(L,X)+P2(L-1,X))
     1-(A(X,Y)+B(X,Y))*(P1(L-1,X)+H*K1A*0.5)
      K3A=B(X,Y+1)*0.5*(P2(L,X)+P2(L-1,X))
     1-(A(X,Y)+B(X,Y))*(P1(L-1,X)+H*K2A*0.5)
      K4A=B(X,Y+1)*P2(L,X)
     1-(A(X,Y)+B(X,Y))*(P1(L-1,X)+H*K3A)
      P1(L,X)=P1(L-1,X)+(H/6.0)*(K1A+2.0*K2A+2.0*K3A+K4A)
      IF(Y.EQ.NO.AND.L.EQ.2)GO TO 601
      GC TO 200
  601 E=ABS(P1(2,MO1)-((B(MO+1,NO+1)*NO/A(MO+1,NO+1))*
     1(EXP((A(MO+1,NO+1)/NO)*H)-1.0)*P2(2,MO1)))
      IF(E.LE.0.001)GO TO 200
      H=H*0.5
      GO TO 600
  504 IF(Y.EQ.NBP+1)GO TO 505
      K1F           =A(X+1,Y)*P1(L-1,X+1)
      K2F=A(X+1,Y)*0.5*(P1(L,X+1)+P1(L-1,X+1))
      K3F=A(X+1,Y)*0.5*(P1(L,X+1)+P1(L-1,X+1))
      K4F=A(X+1,Y)*P1(L,X+1)
      P1(L,X)=P1(L-1,X)+(H/6.0)*(K1F+2.0*K2F+2.0*K3F+K4F)
      GO TO 200
  505 P1(L,X)=0.0
      GC TO 200
  506 K1D=A(X+1,Y)*P1(L-1,X+1)-(A(X,Y)+B(X,Y))*P1(L-1,X)
      K2D=A(X+1,Y)*0.5*(P1(L,X+1)+P1(L-1,X+1))-
     1(A(X,Y)+B(X,Y))*(P1(L-1,X)+H*K1D*0.5)
      K3D=A(X+1,Y)*0.5*(P1(L,X+1)+P1(L-1,X+1))
     1-(A(X,Y)+B(X,Y))*(P1(L-1,X)+H*K2D*0.5)
      K4D=A(X+1,Y)*P1(L,X+1)
     1-(A(X,Y)+B(X,Y))*(P1(L-1,X)+H*K3D)
      P1(L,X)=P1(L-1,X)+(H/6.0)*(K1D+2.0*K2D+2.0*K3D+K4D)
      IF(X.EQ.MO.AND.L.EQ.2)GO TO 602
      GO TO 200
  602 E=ABS(P1(2,MO)-((A(MO+1,NO+1)*MO/B(MO+1,NO+1))*
     1(EXP((B(MO+1,NO+1)/MO)*H)-1.0)*P1(2,MO1)))
      IF(E.LE.0.001)GO TO 200
      H=H*0.5
      GO TO 600
```

```
507 K1G             =B(X,Y+1)*P2(L-1,X)
    K2G=B(X,Y+1)*0.5*(P2(L,X)+P2(L-1,X))
    K3G=B(X,Y+1)*0.5*(P2(L,X)+P2(L-1,X))
    K4G=B(X,Y+1)*P2(L,X)
    P1(L,X)=P1(L-1,X)+(H/6.0)*(K1G+2.0*K2G+2.0*K3G+K4G)
    GO TO 200
509 K1G             =B(X,Y+1)*P2(L-1,X)
    K2G=B(X,Y+1)*0.5*(P2(L,X)+P2(L-1,X))
    K3G=B(X,Y+1)*0.5*(P2(L,X)+P2(L-1,X))
    K4G=B(X,Y+1)*P2(L,X)
    P1(L,X)=P1(L-1,X)+(H/6.0)*(K1G+2.0*K2G+2.0*K3G+K4G)
200 CONTINUE
210 CONTINUE
300 CONTINUE
    DO 1 I=MBP1,MO1
    IF(IT.EQ.0)GO TO 50
    PROB(I-MBP,Y-NBP)=P1(IT,I)
 50 DO 1 J=1,LO1
    P2(J,I)=P1(J,I)
    VAR(J)=VAR(J)+(I-1.0)*(I-1.0)*P1(J,I)
    EST(J)=EST(J)+(I-1.0)*P1(J,I)
  1 CONTINUE
    Z=Y-1
    IF(IT.GT.0)GO TO 901
    IF(MO1-MBP.GT.15)GO TO 901
C
    WRITE(6,100)Z,(I,I=MBP,MO)
C
C
901 DO 99 J=1,LO1
    D(J)=(J-1)*H
C
    IF(IT.GT.0)GO TO 99
    IF(MO1-MBP.GT.15)GO TO 99
    WRITE(6,101)D(J),(P1(J,I),I=MBP1,MO1)
C
 99 CONTINUE
400 CONTINUE
    TI=(IT-1)*H
    WRITE(6,105)H
    WRITE(6,103)
    A2=SQRT(A1*B1)
    A3=SQRT(A1/B1)
    A4=1.0/A3
    A5=MO-NO*A3
    A6=MO+NO*A3
    DO 2 J=1,LO1
    LC=J
    F1=EXP(A2*D(J))*A5
    F2=EXP(-A2*D(J))*A6
    DETERM(J)=0.5*(F1+F2)
    VAR(J)=VAR(J)-EST(J)*EST(J)
    WRITE(6,102)D(J),EST(J),DETERM(J),VAR(J)
    IF(DETERM(J).LE.MBP)GO TO 850
    YFORCE=0.5*A4*(F2-F1)
    IF(YFORCE.LE.NBP)GO TO 851
C
  2 CONTINUE
    GO TO 700
850 IW=1
    GO TO 700
851 IW=0
    GO TO 700
700 IF(IT.EQ.0)GO TO 900
    WRITE(6,106)TI
    WRITE(6,107)(NT,NT=NBP,NO)
    MJ=MO1-MBP
    NJ=NO1-NBP
    DO 110 I=1,MJ
    MT=I+MBP-1
110 WRITE(6,104)MT,(PROB(I,J),J=1,NJ)
900 IF(IT.GT.0)GO TO 902
```

```
      DMIN=DETERM(LC)
      TF1=0.25*D(LC)/H
      TF2=2.0*TF1
      TF3=3.0*TF1
      WRITE(6,108)MO,NO,MBP,NBP,D(LC),TF1,TF2,TF3
      IF(IW.EQ.1)GO TO 852
      WRITE(6,109)
      GO TO 853
  852 WRITE(6,111)
  853 IF(EST(LC).LT.DETERM(LC))GO TO 270
      DM1=MO-DMIN
      DO 903 J=1,LC
      DETERM(J)=((DETERM(J)-DMIN)/DM1)*100.0
      D(J)=D(J)*100.0/D(LC)
  903 CONTINUE
      CALL PLOTS
      CALL SCALE(DETERM,LC,1,9.0,1.0,YMIN,DY)
      CALL SCALE(D,LC,1,6.0,1.0,XMIN,DX)
      CALL AXIS(0.0,0.0,XTITLE,-4,6.0,0.0,XMIN,DX)
      CALL AXIS(0.0,0.0,YTITLE,4,9.0,90.0,YMIN,DY)
      CALL LINE(D,DETERM,LC,1,1)
      DO 14 I=1,LC
      EST(I)=(((EST(I)-DMIN)/DM1)*100.0-YMIN)/DY
   14 CONTINUE
      CALL LINE(D,EST,LC,1,1)
      CALL PLOTE
      GO TO 271
  270 DM1=MO-DMIN
      DO 950 J=1,LC
      DETERM(J)=((DETERM(J)-DMIN)/DM1)*100.0
      D(J)=D(J)*100.0/D(LC)
  950 CONTINUE
      CALL PLOTS
      CALL SCALE(DETERM,LC,1,9.0,1.0,YMIN,DY)
      CALL SCALE(D,LC,1,6.0,1.0,XMIN,DX)
      CALL AXIS(0.0,0.0,ZTITLE,-4,6.0,0.0,XMIN,DX)
      CALL AXIS(0.0,0.0,YTITLE,4,9.0,90.0,0.0,DY)
      CALL LINE(D,DETERM,LC,1,1)
      DO 30 I=1,LC
      EST(I)=(((EST(I)-DMIN)/DM1)*100.0-YMIN)/DY
   30 CONTINUE
      CALL LINE(D,EST,LC,1,1)
      CALL PLOTE
C
  271 IF(IT.EQ.0)GO TO 999
  902 DO 15 K=1,MS
   15 EST(K)=K-1
      DO 16 K=1,NS
   16 DETERM(K)=K-1
      AA=15.0
      BB=30.0
      PMAX=PROB(1,1)
      DO 18 I=1,MS
      DO 18 J=1,NS
      IF(PROB(I,J).GT.PMAX)PMAX=PROB(I,J)
   18 CONTINUE
      CONST=8.0/PMAX
      DO 19 I=1,MS
      DO 19 J=1,NS
      PROB(I,J)=CONST*PROB(I,J)
   19 CONTINUE
      CALL PLT3D1(EST,MS,DETERM,NS,PROB,AA,BB,F,TTL,SIZE,
     1WK,IDN,KX,KY,NKXY,0)
  999 CONTINUE
  100 FORMAT(1X////,3X,'THE FOLLOWING ARE FOR ',I3,' SURVI',
     1'VORS ON THE Y SIDE'///,20X,'NO. OF X SURVIVORS'//,
     26X,'TIME',15I7////)
  101 FORMAT(6X,F7.3,15F7.4)
  102 FORMAT(15X,F7.3,15X,F11.7,15X,F11.7,18X,F11.7)
  103 FORMAT(15X,' TIME  ',15X,'EXPECTED NO. OF SURVIVORS',
     15X,'DETERMINISTIC EQUIVALENT',5X,'VARIANCE'/)
  104 FORMAT(13X,I7,15F7.4//)
```

```
105 FORMAT(1X///,5X,'H IS ',F11.7////)
106 FORMAT(1X,/////,3X,'MATRIX OF PROBABILITIES OF NUMBER'
   1' OF SURVIVORS AT TIME ',F11.7//)
107 FORMAT(19X,15I7//)
108 FORMAT(1X///,5X,'INITIAL FORCE SIZE FOR X IS ',I3/,
   15X,'INITIAL FORCE SIZE FOR Y IS ',I3/,5X,'BREAKPOINT',
   2' FORCE LEVEL FOR X IS ',I3/,5X,'BREAKPOINT FORCE ',
   3'LEVEL FOR Y IS ',I3/,5X,'BATTLE WILL TERMINATE AT ',
   4'TIME ',F11.7///,5X,'TO COMPUTE 3-D GRAPHS AT 25, ',
   5'50, AND 75% TIMES INTO THE BATTLE,IT SHOULD BE'/,10X,
   6'FOR 25%',F11.7/,10X,'FOR 50%',F11.7/,10X,'FOR 75%',F1
109 FORMAT(5X,'THE WINNER IS X'////)
111 FORMAT(5X,'THE WINNER IS Y'////)
    END
    FUNCTION A(I,J)
    A=0.008*(J-1.0)
    RETURN
    END
    FUNCTION B(I,J)
    B=0.004*(I-1.0)
    RETURN
    END
```

```
C                             COMPUTER PROGRAM
C
C
C     SOLUTION TO SPRINGALLS RECURSIVE SOLUTION FOR THE
C        PROBABILITY ONE SIDE WILL WIN
C
C
      DIMENSION X(150,40),NS(5),RS(8)
      NS(1)=4
      NS(2)=5
      NS(3)=10
      NS(4)=20
      NS(5)=40
      RS(1)=0.25
      RS(2)=0.5
      RS(3)=1.0
      RS(4)=2.0
      RS(5)=3.0
      RS(6)=4.0
      RS(7)=5.0
      RS(8)=6.0
      FX=1.2
      FY=1.2
      DO 8 KI=1,5
      FX=FX-0.2
      FY=FY-0.2
      DO 6 L=1,5
      NO=NS(L)
      DO 7 K=1,8
      R=RS(K)
      SM=2.0*NO*NO*(1.0-(1.0-FY)*(1.0-FY))*R/(1.0-(1.0-FX)
     1*(1.0-FX))
      MO=SQRT(SM)
      MBP=(1.0-FX)*MO
      NBP=(1.0-FY)*NO
      MBP1=MBP+1
      NBP1=NBP+1
      MDIFF=MO-MBP
      NDIFF=NO-NBP
      MCOUNT=MDIFF-1
      NCOUNT=NDIFF-1
      X(MDIFF,NDIFF)=1.0/(A(MO,NO)+B(MO,NO))
      DO 1 J=1,NCOUNT
      NREAL=NO-J
      NSTORE=NDIFF-J
      X(MDIFF,NSTORE)=(B(MO,NREAL+1)*X(MDIFF,NSTORE+1))/
     1(A(MO,NREAL)+B(MO,NREAL))
    1 CONTINUE
      DO 2 I=1,MCOUNT
      MREAL=MO-I
      MSTORE=MDIFF-I
      X(MSTORE,NDIFF)=(A(MREAL+1,NO)*X(MSTORE+1,NDIFF))/
     1(A(MREAL,NO)+B(MREAL,NO))
      DO 3 J=1,NCOUNT
      NREAL=NO-J
      NSTORE=NDIFF-J
      X(MSTORE,NSTORE)=(A(MREAL+1,NREAL)*X(MSTORE+1,NSTORE)
     1+B(MREAL,NREAL+1)*X(MSTORE,NSTORE+1))/
     2(A(MREAL,NREAL)+B(MREAL,NREAL))
    3 CONTINUE
    2 CONTINUE
      PMWIN=0.0
      PNWIN=0.0
      DO 4 I=1,MDIFF
      PMWIN=PMWIN+B(I+MBP,NBP1)*X(I,1)
    4 CONTINUE
      DO 5 J=1,NDIFF
      PNWIN=PNWIN+A(MBP1,J+NBP)*X(1,J)
    5 CONTINUE
      SUM=PMWIN+PNWIN
      S=MO*MO*(1.-(1.-FX)*(1.-FX))/(2.*(1.-(1.-FY)*(1.-FY))
     1*NO*NO)
```

```
      WRITE(6,100)MO,MBP,NO,NBP,FX,FY
      WRITE(6,101)PMWIN,PNWIN,SUM,S
    7 CONTINUE
    6 CONTINUE
    8 CONTINUE
  100 FORMAT(5X,'MO=',I4,5X,'MBP=',I4,5X,'NO=',I4,5X,'NBP=',
     1I4,5X,'FX=',F6.2,5X,'FY=',F6.2/)
  101 FORMAT(5X,'PMWIN=',F11.7,5X,'PNWIN=',F11.7,5X,'SUM=',
     1F11.7,5X,'R=',F11.7///)
      END
      FUNCTION A(I,J)
      A=0.008*J
      RETURN
      END
      FUNCTION B(I,J)
      B=0.004*I
      RETURN
      END
```

BIBLIOGRAPHY

1. Adkins, Richard C., Analysis of Unit Breakpoints in Land Combat, M.S. Thesis, Naval Postgraduate School, March 1975.

2. Bach, R.E., Dolansky, L., and Stubbs, H.L., "Some Recent Contributions to the Lanchester Theory of Combat," Operations Research, v. 10, p. 314-326, 1962.

3. Bishop, Albert B., and Clark, Gordon M., The Tank Weapons System, Systems Research Group, Department of Industrial Engineering, The Ohio State University, 30 June 1969.

4. Bonder, Seth, "An Overview of Land Battle Modeling in the U.S.", Proceedings, Thirteenth Annual U.S. Army Operations Research Symposium, p. 73-88, 29 October - 1 November 1974.

5. Bostick, Steven P., Brandi, Francis X., Burnham, C. Alan, and Hurt, James J., "The Interface Between DYNTACS-X and Bonder I.U.A.", Proceedings, Thirteenth Annual U.S. Army Operations Research Symposium, p. 494-502, 29 October - 1 November 1974.

6. Brooks, F., "The Stochastic Properties of Large Scale Battle Models", Operations Research, v. 13, p. 1-17, 1965.

7. Brown, Richard H., "Theory of Combat: The Probability of Winning", Operations Research, v. 11, p. 418-425, 1963.

8. Clark, Gordon M., The Combat Analysis Model, PH.D. Dissertation, The Ohio State University, 1969.

9. Dolansky, L., "Present State of the Lanchester Theory of Combat", Operations Research, v. 12, p. 344-358, 1964.

10. Engineer Strategic Studies Group, Bibliography of Publications, 1972.

11. General Research Corporation, Operations Analysis Division, Gaming and Simulation Department, A Hierarchy of Combat Analysis Models.

12. Hannah, William P., Further Comparisons of Stochastic and Deterministic Models for the Optimal Control of Lanchester-Type Attrition Processes, M.S. Thesis, Naval Postgraduate School, March 1975.

13. Helmbold, R., Decision in Battle: Breakpoint Hypothesis and Engagement Termination Data, The RAND Corporation Report R-772-PR, June 1971.

14. Isbell, J.R., and Marlow, W.H., "Attrition Games", Naval Research Logistics Quarterly, v. 3, p. 71-94, 1956.

15. Jain, J.C., and Nagabhushanam, A., "A Two-State Markovian Correlated Combat", Operations Research, v. 22, p. 440-444, 1974.

16. Johnson, Patricia C., Plotting Package for NPS IBM 360/67, Naval Postgraduate School TN No. 0211-03, January 1974.

17. Kisi, T., and Hirose, T., "Winning Probability in an Ambush Engagement", Operations Research, v. 14, p. 1137-1138, 1966.

18. Koopman, Bernard O., "A Study of the Logical Basis of Combat Simulation", Operations Research, v. 18, p. 855-882, 1970.

19. Lanchester, F.W., "Aircraft in Warfare: The Dawn of the Fourth Arm-No. V., The Principle of Concentration", Engineering, v. 98, p. 422-423, 1914 (reprinted in World of Mathematics, v. IV, p. 2138-2148, J. Newman, ed., Simon and Schuster, New York, 1956).

20. Lee, Woo Young, and Wannasilpa, Amnuay, Comparison of a Deterministic and a Stochastic Model for the Probability of Winning in a Two-Sided Combat Situation, M.S. Thesis, Naval Postgraduate School, September 1972.

21. McCracken, Daniel D., and Dorn, William S., Numerical Methods and FORTRAN Programming, John Wiley and Sons, Inc., 1964.

22. Morse, P.M., and Kimball, G.E., Methods of Operations Research, Chapman and Hall, London, 1950.

23. Naddor, Eliezer, "Dimensions in Operations Research", Operations Research, v. 14, p. 508-514, 1966.

24. Powers, Robert L., and Taylor, James G., Comparison of a Deterministic and a Stochastic Formulation for the Optimal Control of a Lanchester-Type Attrition Process, Paper presented at the International Symposium on Applications of Computers and Operations Research to Problems of World Concern, Washington, D.C., 20-21 August 1973.

25. Raney, Sharon D., PLT3D1: Three Dimensional Isometric or Perspective Off-Line Plotting Subprogram with Hidden Line Elimination, Naval Postgraduate School Technical Memorandum, February 1974.

26. Smith, David G., "The Probability Distribution of the Number of Survivors in a Two-Sided Combat Situation", Operational Research Quarterly, v. 16, No. 4, p. 429-437, 1965.

27. Snow, R.N., Contributions to Lanchester Attrition Theory, The RAND Corporation Report RA-15078, April 1948.

28. Springall, Anthony, Contributions to Lanchester Combat Theory, PH. D. Thesis, Virginia Polytechnic Institute, March 1968.

29. Taylor, James G., Lanchester-Type Models of Warfare, Tutorial delivered at the Thirty-fifth Military Operations Research Symposium, U.S. Naval Academy, Annapolis, Maryland, 1-3 July 1975.

30. Taylor, James G., "Solving Lanchester-Type Equations for 'Modern Warfare' with Variable Coefficients", Operations Research, v. 22, p. 756-770, 1974.

31. Taylor, James G., "Survey on the Optimal Control of Lanchester-Type Attrition Processes", presented at The Symposium on the State-of-the-Art of Mathematics in Combat Models (also Technical Report NPS55Tw70431, Naval Postgraduate School, Monterey, California, March 1974).

32. Taylor, James G., and Parry, Samuel H., "Force Ratio Considerations for Some Lanchester-Type Models of Warfare", Operations Research, v. 23, p. 522-533, 1975.

33. U.S. Army Concepts Analysis Agency, Methodology and Resource Directorate, Tabulation of Models of Interest to USACAA, June 1975.

34. Vennard, John K., Elementary Fluid Mechanics, 3rd ed., John Wiley and Sons, Inc., 1954.

35. Weiss, George H., "A Comparison of a Deterministic and a Stochastic Model for Interaction between Antagonistic Species", Biometrics, v. 17, p. 595-602, 1963.

36. Weiss, Herbert K., "Lanchester-Type Models of Warfare", Proceedings, First International Conference on Operational Research, p. 82-99, Oxford, September 1957.

37. Willard, D., Lanchester as Force in History: An Analysis of Land Battles of the Years 1618-1905, Research Analysis Corporation Report RAC-TP-75, November 1962.

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